**SOLUTION FO SAT-1**
SEMESTER-II (GENERAL)- 2020
Subject: Mathematics
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Teacher: Dr. Prasanta Paul

1. Solve the IVP $(x^2+1)\frac{dy}{dx}+4xy=x, \quad y(2)=1.$

1. IVP $(x^2+1)\frac{dy}{dx}+4xy=x$, $y(2)=1$

$\therefore \quad \frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{x}{x^2+1} \quad \text{—(1)}$

I.F. is $e^{\int \frac{4x}{x^2+1}dx} = e^{2\log(x^2+1)} = (x^2+1)^2$

Multiplying by integrating factor with given eqn (1) and on integration we get

$$y(x^2+1)^2 = \int \frac{x}{x^2+1}(x^2+1)^2\,dx + C$$
$$= \int x(x^2+1)\,dx + C$$
$$= \frac{x^4}{4}+\frac{x^2}{2}+C.$$
$$= \frac{x^2}{4}(x^2+2)+C$$

Given that at $x=2$, $y=1$

$\therefore \quad 1\cdot(4+1)^2 = \frac{4}{4}(4+2)+C$

$\Rightarrow \quad 25 = 6+C \Rightarrow C = 25-6 = 19$

Thus the soln is $y(x^2+1)^2 = \frac{x^2}{4}(x^2+2)+19$ Ans.

2. Given that $y=x+1$ is a solution of $(x+1)^2\frac{d^2y}{dx^2}-3(x+1)\frac{dy}{dx}+3y=0$. Find the general solution.

2. First we observe that $y=1+x$ satisfy given equn. Then let

$y=(1+x)v$

Using the above transformation we want to reduce the given ~~diff. equ~~ linear equation to the first order homogeneous diff. equation as below:

$\frac{dy}{dx}=(1+x)\frac{dv}{dx}+v$, and $\frac{d^2y}{dx^2}=(1+x)\frac{d^2v}{dx^2}+2\frac{dv}{dx}$

Substituting the expressions for $y$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ into the equn

$(x+1)^2\frac{d^2y}{dx^2}-3(x+1)\frac{dy}{dx}+3y=0$ we get

$(x+1)^2\left[(1+x)\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right]-3(x+1)\left[(1+x)\frac{dv}{dx}+v\right]+3(1+x)v=0$

$\Rightarrow \quad (x+1)^3\frac{d^2v}{dx^2}-(x+1)^2\frac{dv}{dx}=0$

$\Rightarrow \quad (x+1)\frac{d^2v}{dx^2}-\frac{dv}{dx}=0$

Letting $w = \frac{dv}{dx}$ we obtain the first-order homogeneous linear diff. eqn.

$$(x+1)\frac{dw}{dx} - w = 0$$

$$\Rightarrow \frac{dw}{w} = \frac{dx}{x+1}$$

Integrating, we obtain the general solⁿ

$$w = c\,(x+1)$$

Choosing $c=1$, we recall that $w = \frac{dv}{dx}$ and integrating to obtain the function $v$ given by $v = \frac{(x+1)^2}{2}$

Now $y = (1+x)v = (1+x)\frac{(x+1)^2}{2} = \frac{1}{2}(x+1)^3$

is another solⁿ of the given diff. eqⁿ.

The solⁿs $y = (1+x)$ and $y = \frac{1}{2}(x+1)^3$ are linearly independent.

Hence the general solⁿ of the given diff. eqⁿ is

$$y = c_1(1+x) + c_2\frac{1}{2}(x+1)^3 \quad \text{Ans.}$$

3. Solve by method of variation of parameters: $(D^2+4)y = \operatorname{cosec} 2x$, where $D \equiv \frac{d}{dx}$.

3. $(D^2+4)y = \operatorname{cosec} 2x$, $D \equiv \frac{d}{dx}$.

To find C.F. Auxiliary equation is $m^2+4=0$ gives $m = \pm 2i$

The C.F. is $c_1\cos 2x + c_2\sin 2x$.

To find P.I. Let $y_1 = \cos 2x$, $y_2 = \sin 2x$

The Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0$$

Therefore $y_1$ & $y_2$ are independent solutions.

Let P.I. be $y_p = v_1(x)\cos 2x + v_2(x)\sin 2x$

$\therefore Dy_p = -2v_1\sin 2x + 2v_2\cos 2x + v_1'\cos 2x + v_2'\sin 2x$

choose $v_1$ & $v_2$ such that $v_1'\cos 2x + v_2'\sin 2x = 0$ —— ①

so that, $Dy_p = -2v_1\sin 2x + 2v_2\cos 2x$

& $D^2y_p = -4v_1\cos 2x - 4v_2\sin 2x - 2v_1'\sin 2x + 2v_2'\cos 2x$

$= -4y_p - 2v_1'\sin 2x + 2v_2'\cos 2x$

$\Rightarrow D^2y_p + 4y_p = -2v_1'\sin 2x + 2v_2'\cos 2x$

$\Rightarrow -2v_1'\sin 2x + 2v_2'\cos 2x = \operatorname{cosec} 2x$ —— ②

Solving ① and ② we get

$$v_1' = \frac{-\sin 2x \cdot \operatorname{cosec} 2x}{2} = -\frac{1}{2} \Rightarrow v_1 = -\frac{1}{2}x$$

and $v_2' = \dfrac{\cos x \operatorname{cosec} 2x}{2} = \frac{1}{2}\cot 2x$

$\Rightarrow v_2 = \frac{1}{4}\log|\sin 2x|$

Accordingly, $y_p = -\frac{1}{2}x\cos 2x + \frac{1}{4}\log|\sin 2x|\ \sin x$.

$\therefore$ the complete solution is

$y = c_1\cos 2x + c_2\sin 2x - \frac{x}{2}\cos x + \frac{1}{4}\log|\sin x|\ \sin 2x$ Ans

4. Solve, $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$, where $D \equiv \frac{d}{dx}$.

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$(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$, $D \equiv \frac{d}{dx}$

We put $x = e^t$ or $t = \log x$.

If the operator $x\frac{d}{dx} = xD$ be denoted by $D_1$, then we get $\left[D_1 = \frac{d}{dt}\right]$

$xDy = D_1 y$; $x^2D^2y = D_1(D_1-1)y$.

The given eqn reduces to an equation with constant coefficients with independent variable $t$ in place of $x$; thus

$\{D_1(D_1-1) - D_1 + 4\}y = \cos t + e^t\sin t$

$\Rightarrow \{D_1^2 - 2D_1 + 4\}y = \cos t + e^t\sin t$

$m^2 - 2m + 4 = 0$

$\Rightarrow m = \frac{2\pm\sqrt{4-16}}{2}$

$= \frac{2\pm2\sqrt{3}i}{2}$

$= 1\pm\sqrt{3}i$

The C.F. is $e^t(C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t)$

$= x\left[C_1\cos(\sqrt{3}\log x) + C_2\sin(\sqrt{3}\log x)\right]$, $C_1, C_2$ are arbitrary constants.

and P.I. is $\dfrac{1}{D_1^2-2D_1+4}\left[\cos t + e^t\sin t\right]$

$= \dfrac{1}{D_1^2-2D_1+4}\cos t + \dfrac{1}{D_1^2-2D_1+4}e^t\sin t$

$= \dfrac{1}{-1^2-2D_1+4}\cos t + e^t\dfrac{1}{(D_1+1)^2-2(D_1+1)+4}\sin t$

$= \dfrac{1}{3-2D_1}\cos t + e^t\dfrac{1}{D_1^2+3}\sin t$

$= \dfrac{3+2D_1}{9-4D_1^2}\cos t + e^t\dfrac{1}{-1^2+3}\sin t$

$= \dfrac{3+2D_1}{9-4(-1^2)}\cos t + \dfrac{e^t\sin t}{2}$

$= \frac{1}{13}\left[3\cos t - 2\sin t\right] + \frac{1}{2}e^t\sin t$

$= \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{1}{2}x\sin(\log x)$

$\therefore$ the complete primitive is

$y = x\left[C_1\cos(\sqrt{3}\log x) + C_2\sin(\sqrt{3}\log x)\right] + \frac{3}{13}\cos(\log x) - \frac{2}{13}\sin(\log x)$
$+ \frac{1}{2}x\sin(\log x)$ Ans:

5. Solve the PDE by Lagrange's Method: $px(x+y) - qy(x+y) + (x-y)(2x+2y+z) = 0$.

5. The given eq$^n$ may be written as ~~$x(x+y)p - y(x+y)q = -(x-y)(2x+2y+$~~

$x(x+y)p - y(x+y)q = -(x-y)(2x+2y+z)$

Lagrange's subsidiary equations are

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)} \quad \text{—(1)}$$

first two ratios of (1) at once give, on integration $xy = c_1$.

Again each fraction of (1) becomes

$$= \frac{dx+dy}{x^2-y^2} = \frac{dx+dy+dz}{x(x+y) - y(x+y) - (x-y)(2x+2y+z)}$$

or $$\frac{d(x+y)}{(x-y)(x+y)} = \frac{d(x+y+z)}{(x-y)(x+y-2x-2y-z)}$$

$\Rightarrow$ On integration we get $(x+y)\cdot(x+y+z) = c_2$

$\therefore$ the required general sol$^n$ is $\phi(xy, (x+y)(x+y+z)) = 0$, where $\phi$ is an arbitrary function. Ans