**SOLUTION OF SAT-1**

SEMESTER-II, 2020

Subject: Mathematics

Course Code: MTMACORE04T

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1. a) Solve, using the method of undetermined coefficients: $(D^2 - 3D + 2)y = 14\sin 2x - 18\cos 2x$.

1. a) $(D^2-3D+2)y = 14\sin 2x - 18\cos 2x$

To find C.F.: let $y = e^{mx} \neq 0$ be a trial soln, then the auxiliary equn is $m^2-3m+2=0$ whereby $m = \frac{3\pm\sqrt{9-8}}{2} = 1, 2$.

C.F. is $(c_1 e^{x} + c_2 e^{2x})$.

To find P.I.:

Since $\sin 2x$, $\cos 2x$ are not appear in C.F., hence we assume $y_p = A\sin 2x + B\cos 2x$

Then $Dy_p = 2A\cos 2x - 2B\sin 2x$, $D^2 y_p = -4A\sin 2x - 4B\cos 2x$ and from $D^2y_p - 3Dy_p + 2y_p = 14\sin 2x - 18\cos 2x$, we have

$-4A\sin 2x - 4B\cos 2x - 6A\cos 2x + 6B\sin 2x + 2A\sin 2x + 2B\cos 2x = 14\sin 2x - 18\cos 2x$

$\Rightarrow 4A\sin 2x - 8B\cos 2x = 14\sin 2x - 18\cos 2x$

Comparing both sides we get

$4A = 14 \Rightarrow A = 7/2$ & $-8B = -18 \Rightarrow B = \frac{9}{4}$

$\therefore y_p = \frac{7}{2}\sin 2x + \frac{9}{4}\cos 2x$.

Hence, the general soln is $y = c_1 e^{x} + c_2 e^{2x} + \frac{7}{2}\sin 2x + \frac{9}{4}\cos 2x$ Ans

1. b) Solve, $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$, where $D \equiv \frac{d}{dx}$.

1. b) $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$, $D \equiv \frac{d}{dx}$

We put $x = e^t$ or $t = \log x$.

If the operator $x\frac{d}{dx} = xD$ be denoted by $D_1$, then we get $\left[D_1 = \frac{d}{dt}\right]$

$xDy = D_1 y$; $x^2D^2y = D_1(D_1-1)y$.

The given equn reduces to an equation with constant coefficients with independent variable $t$ in place of $x$; thus

$\{D_1(D_1-1) - D_1 + 4\}y = \cos t + e^t \sin t$

$\Rightarrow \{D_1^2 - 2D_1 + 4\}y = \cos t + e^t\sin t$

$m^2 - 2m + 4 = 0$
$\Rightarrow m = \frac{2\pm\sqrt{4-16}}{2} = \frac{2\pm 2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$

The C.F. is $e^{t}(c_1\cos\sqrt{3}t + c_2\sin\sqrt{3}t)$

$= x\left[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)\right]$, $c_1, c_2$ are arbitrary constants.

and P.I. is $\frac{1}{D_1^2-2D_1+4}\left[\cos t + e^t\sin t\right]$

$= \frac{1}{D_1^2-2D_1+4}\cos t + \frac{1}{D_1^2-2D_1+4}e^t\sin t$

$$= \frac{1}{-1^2-2D_1+4}\cos t + e^t \frac{1}{(D_1+1)^2-2(D_1+1)+4}\sin t$$

$$= \frac{1}{3-2D_1}\cos t + e^t \frac{1}{D_1^2+3}\sin t$$

$$= \frac{3+2D_1}{9-4D_1^2}\cos t + e^t \frac{1}{-1^2+3}\sin t$$

$$= \frac{3+2D_1}{9-4(-1)}\cos t + \frac{e^t \sin t}{2}$$

$$= \frac{1}{13}[3\cos t - 2\sin t] + \frac{1}{2}e^t \sin t$$

$$= \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{1}{2}x\sin(\log x)$$

$\therefore$ the complete primitive is

$$y = x[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)] + \frac{3}{13}\cos(\log x) - \frac{2}{13}\sin(\log x) + \frac{1}{2}x\sin(\log x) \quad \underline{\text{Ans:}}$$

2. a) Solve by method of variation of parameters: $(D^2 + a^2)y = \tan ax$.

<u>SOLUTION</u>: See Diff. Equation Book by Gosh & Maity Page Number 226, Example 5.7.37.

2. b) Solve, using the method of undetermined coefficients: $(D^2 + 4)y = x^2 \sin 2x$, where $D \equiv \frac{d}{dx}$.

<u>SOLUTION</u>: See Diff. Equation Book by Gosh & Maity Page Number 210, Example 5.7.27.

3. a) Solve, $(x^2D^2 - 3xD + 5)y = x^2 \sin(\log x)$, where $D \equiv \frac{d}{dx}$.

3.a) $(x^2D^2 - 3xD + 5)y = x^2\sin(\log x)$, $D \equiv \frac{d}{dx}$

We put $x = e^t$ or $t = \log x$.

If the operator $x\frac{d}{dx}$ be denoted by $D_1$, then we get

$xDy = D_1y$, $x^2D^2y = D_1(D_1-1)y$, $D_1 \equiv \frac{d}{dt}$

The given eq$^n$ reduces to an eq$^n$ with constant coefficients with independent variable $t$ in place of $x$; thus

$\{D_1(D_1-1) - 3D_1 + 5\}y = e^{2t}\sin t$

$\Rightarrow (D_1^2 - 4D_1 + 5)y = e^{2t}\sin t$

Let $y = e^{mt}$ to be trail sol$^n$ of the reduce eq$^n$, then auxiliary eq$^n$ is $m^2 - 4m + 5 = 0 \Rightarrow m = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

C.F. is $e^{2t}(c_1\cos t + c_2\sin t)$, where $t = \log x$, $c_1, c_2$ are arbitrary constants.

$= x^2[c_1\cos(\log x) + c_2\sin(\log x)]$

P.I. is $\frac{1}{D_1^2 - 4D_1 + 5}e^{2t}\sin t$

$$= e^{2t}\frac{1}{(D_1+2)^2 - 4(D_1+2) + 5}\sin t$$

$$= e^{2t} \cdot \frac{1}{D_1^2+4D_1+4-4D_1-8+5} \sin t$$

$$= e^{2t} \frac{1}{D_1^2+1} \sin t$$

$$= e^{2t} \frac{1}{D_1^2+1} \frac{1}{2i}\left(e^{it}-e^{-it}\right)$$

$$= \frac{e^{2t}}{2i}\left[\frac{1}{D_1^2+1} e^{it} - \frac{1}{D_1^2+1} e^{-it}\right]$$

$$= \frac{e^{2t}}{2i}\left[e^{it} \frac{1}{(D_1+i)^2+1} \cdot 1 - e^{-it} \frac{1}{(D_1-i)^2+1} \cdot 1\right]$$

$$= \frac{e^{2t}}{2i}\left[e^{it} \frac{1}{D_1^2+2iD_1} \cdot 1 - e^{-it} \frac{1}{D_1^2-2iD_1} \cdot 1\right]$$

$$= \frac{e^{2t}}{2i}\left[e^{it} \frac{1}{2iD_1}\left(1-\frac{D_1}{2i}\right) \cdot 1 - \frac{e^{-it}}{-2iD_1}\left(1+\frac{D_1}{2i}\right) \cdot 1\right]$$

$$= \frac{e^{2t}}{2i}\left[\frac{e^{it}}{2i} \frac{1}{D_1} \cdot 1 + \frac{e^{-it}}{2i} \cdot \frac{1}{D_1} \cdot 1\right]$$

$$= \frac{e^{2t}}{2i}\left[\frac{e^{it}}{2i} t + \frac{e^{-it}}{2i} t\right]$$

$$= \frac{t e^{2t}}{2i^2} \frac{1}{2}\left[e^{it}+e^{-it}\right]$$

$$= \frac{t e^{2t}}{-2} \cos t \qquad \left[\because \cos t = \frac{1}{2}\left(e^{it}+e^{-it}\right)\right]$$

$$= -\frac{1}{2} x^2 \log x \cos(\log x)$$

$\therefore$ the general sol$^n$ is

$$y = x^2\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] - \frac{1}{2}x^2 \log x \cos(\log x) \text{ (Ans)}$$

3. b) Solve by method of variation of parameters: $(D^2+4)y = \operatorname{cosec} 2x$, where $D \equiv \frac{d}{dx}$.

3. b) $(D^2+4)y = \operatorname{cosec} 2x$, $D \equiv \frac{d}{dx}$.

To find C.F. Auxiliary equation is $m^2+4=0$ gives $m = \pm 2i$

The C.F. is $c_1 \cos 2x + c_2 \sin 2x$.

To find P.I. Let $y_1 = \cos 2x$, $y_2 = \sin 2x$

The wronskian of $y_1$ and $y_2$ is

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0$$

Therefore $y_1$ & $y_2$ are independent solutions.

Let P.I. be $y_p = v_1(x)\cos 2x + v_2(x) \sin 2x$

$\therefore Dy_p = -2v_1 \sin 2x + 2v_2 \cos 2x + v_1' \cos 2x + v_2' \sin 2x$

choose $v_1$ & $v_2$ such that $v_1' \cos 2x + v_2' \sin 2x = 0$ —— (1)

so that, $Dy_p = -2v_1 \sin 2x + 2v_2 \cos 2x$

& $D^2 y_p = -4v_1 \cos 2x - 4v_2 \sin 2x - 2v_1' \sin 2x + 2v_2' \cos 2x$

$$= -4y_p - 2v_1' \sin 2x + 2v_2' \cos 2x$$

$$\Rightarrow D^2y_p + 4y_p = -2v_1' \sin 2x + 2v_2' \cos 2x$$

$$\Rightarrow -2v_1' \sin 2x + 2v_2' \cos 2x = \operatorname{cosec} 2x \quad \text{—— (2)}$$

solving (1) and (2) we get

$$v_1' = \frac{-\sin 2x \cdot \operatorname{cosec} 2x}{2} = -\frac{1}{2} \Rightarrow v_1 = -\frac{1}{2}x$$

and $$v_2' = \frac{\cos 2x \operatorname{cosec} 2x}{2} = \frac{1}{2}\cot 2x$$

$$\Rightarrow v_2 = \frac{1}{4}\log|\sin 2x|$$

Accordingly, $y_p = -\frac{1}{2}x\cos 2x + \frac{1}{4}\log|\sin 2x| \sin 2x$.

$\therefore$ the complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{x}{2}\cos 2x + \frac{1}{4}\log|\sin 2x| \sin 2x$$ Ans