SEM 4: MACRO

UNIT 1: Solow MODEL

Introduction

Economic growth refers to an increase in the level of aggregate output or income an economy is able to produce in a unit of time.

Assumptions

1. Capital and labour are the only factors of production.
2. The economy is assumed to be a closed economy without government.
3. Output in any period is net of depreciation.
4. Aggregate production function is linearly homogeneous i.e. has constant returns to scale.
5. There is perfect substitutability between capital and labour.
6. Both factors have positive but diminishing marginal products, i.e. $MP_{K,}$ MP_L>0 while $\frac{\partial MP_K}{\partial K}$,$\frac{\partial MP_L}{\partial L} < 0$. Solow made this assumption as a necessary condition for existence of a unique and stable equilibrium.
7. Population is growing exponentially at a constant rate of growth, i.e. 'n' [$\frac{1}{L}\frac{dL}{dt} = n$]. Therefore the labour supply curve is given by $L = L_0 e^{nt}$. The constant growth rate of labour is given exogenously to the Solow's model. It is called the insttntenous growth rate of labour force.
8. Stock of capital depriciate at a constant rate, say δ. Thus depriciation = δK
9. Economy in every period saves a constant proportion of that period's output, 's'.
10. Investment planned is always equal to savings planned. Thus Solow does away with the short run Keynesian framework since according to him the neoclassical model is best to analyse growth which is inevitably a long run phenomenon.
11. There is always full employment of factors of production.
12. There are no adjustment lags in any market in the economy.

Structure of Solow Model

The aggregate production function is given by $Y = F(K, L)$. Since Solow assumes constant returns to scale, so $\lambda Y = F(\lambda K, \lambda L)$ where λ≥0

If we choose λ= 1/L then we can write the above production function as $\frac{Y}{L} = F(\frac{K}{L}, 1)$. Now if we define $\frac{K}{L}$ as k and $\frac{Y}{L}$ as y then the aggregate production can be rewritten as $\boldsymbol{y} = \boldsymbol{f}\ (\boldsymbol{k})$. <u>This is said</u>

to be the per capita production function where y is the per capita output or income and k is the per capita capital. Thus k is the primary variable in solow model.

Solow assumed such fine properties of the production function in order to ensure the existence of unique stable equilibrium in his model. The following conditions are known as **Inada** conditions:

1. $f'(k) > 0$ for all k

2. $f''(k) < 0$ for all k

3. $\lim_{k\to 0} f'(k) = \infty$

All these properties are satisfied by Cobb-Douglas production function.

Concept of Steady state

Net investment is defined as addition to capital stock over time. Thus net investment = $\frac{dK}{dt} = \dot{K}$

From NI accounting we know, NI= GI- depriciation

Again from saving investment identity we can write, Savings = GI

Therefore, net investment = savings – depriciation

$$\Rightarrow \dot{K} = sY - \delta K$$

$$\Rightarrow \frac{\dot{K}}{K} = \frac{sLf(k)}{K} - \delta$$

$$\Rightarrow \frac{\dot{K}}{K} = \frac{sf(k)}{\frac{K}{L}} - \delta$$

$$\Rightarrow \frac{\dot{K}}{K} = \frac{sf(k)}{k} - \delta \text{........................ (1)}$$

We know $k = \frac{K}{L}$

Applying logarithm we have

$$\log k = \log K - \log L$$

Differentiating w.r.t. 't' we get,

$$\frac{1}{k}\frac{dk}{dt} = \frac{1}{K}\frac{dK}{dt} - \frac{1}{L}\frac{dL}{dt}$$

i.e., $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$

$$\Rightarrow \frac{\dot{k}}{k} = \frac{sf(k)}{k} - \delta - n \text{ (from 1)}$$

$$\Rightarrow \dot{\mathbf{k}} = \mathbf{sf}(\mathbf{k}) - (\boldsymbol{\delta} + \mathbf{n})k \text{ .... (2)}$$

(2) is the basic differential equation of solow model.

The capital per capita that remains constant in the very long run in the Solow's model is known as the Steady-state capital stock per capita. It is denoted as k^*. Here the steady state is defined as the state of the economy which when reached will continue. In such a state, also known as the balanced growth path, the ratio of capital to output also remains constant., i.e., $\dot{k} = 0$

Therefore, $\dot{\mathbf{k}} = \mathbf{sf}(\mathbf{k}) - (\boldsymbol{\delta} + \mathbf{n})\mathrm{k} = 0$

$\Rightarrow \mathbf{sf}(\mathbf{k}) = (\boldsymbol{\delta} + \mathbf{n})\mathrm{k}$ →Condition for steady-state

We know, $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$

At steady state, $\dot{k} = 0 \Rightarrow \frac{\dot{K}}{K} = \frac{\dot{L}}{L} = n$

Thus at steady-state growth of per capita capital exactly matches the exogeneously given growth rate of labour force

Again we know,

$$Y = Lf(k)$$

$\Rightarrow \log Y = \log L + \log f(k)$ (by taking log on both sides of the equation)

Differentiating w.r.t. 't' we get,

$$\frac{1}{Y}\frac{dY}{dt} = \frac{1}{L}\frac{dL}{dt} - \frac{1}{f(k)}\frac{df(k)}{dt}$$

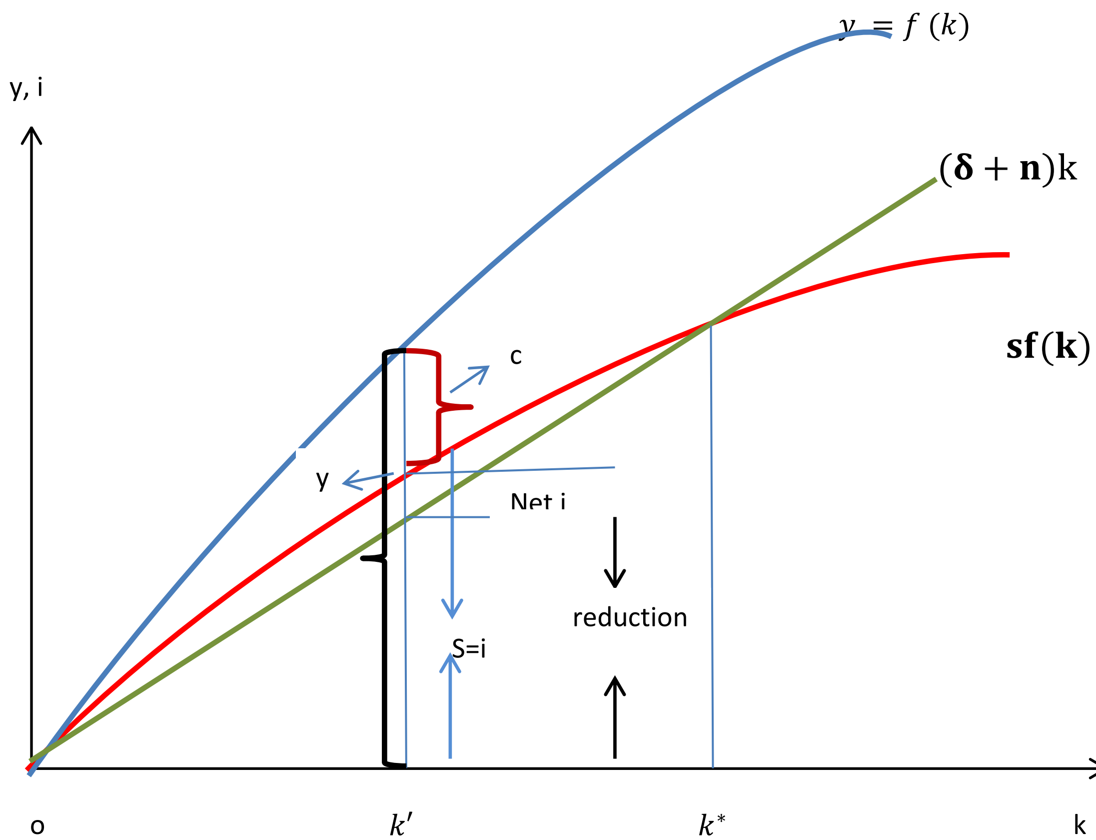
Thus we have, $\frac{\dot{Y}}{Y} = \frac{\dot{L}}{L} = n$ [since, at steady state 'k' is constant so f(k) is also constant. The derivative of f(k) w.r.t. will be zero]

Thus we see that the instantaneous growth rate of output exactly matches the exogeneously given growth rate of labour force.

Diagramatic representation of steady-state

The production function $y = f(k)$ is concave to the horizontal axis by assuming diminishing returns to factor. In the figure, the investment per capita function is drawn as the red coloured concave curve $\mathrm{sf(k)}$. Given the production function, position of gross investment curve is dependent on the exogenously given savings ratio. A higher savings ratio raises the curve upwards and as $s \rightarrow 1$, the investment per capita curve approaches the output per capita curve shown in blue. The green coloured straight line is the reduction in capital curve due to growth of population and depreciation.

At any given capital-output ratio (say k'), the difference in the height of the output per capita curve and investment per capita curve tells us the consumption per capita whereas the difference in the height of the investment per capita and the reduction in capital per capita line tells us the net investment per capita in the economy. If net investment per capita is positive, which happens for all positive capital-labour ratios less than k^*, capital stock per capita in the economy grows. If net investment per capita is negative then capital stock per capita in the economy falls. This occurs at capital-labour ratios more than k^*. Only at k^* , addition and reduction to capital per capita offset each other and therefore *k* eventually remains steady at k^*.



Golden Rule Steady State

The above analysis tells us that given the rate of population growth and rate of depreciation, all possible capital stock per capital can be maintained steadily by choosing an appropriate savings ratio in the economy.

There exists a particular rate of savings and in turn a particular steady state capital stock per capita which maximizes consumption per capita in the steady state. Such a rate of savings is called the **golden rule** level of savings and denoted as s_{gold}.

Derivation of Golden Rule Steady State Condition

From NI accounting we know,

Y= C+ S

$\Rightarrow \frac{Y}{L} = \frac{C}{L} + \frac{S}{L}$ $\Rightarrow y = c + s.y$ $\Rightarrow c = y - s.y$ $\Rightarrow c = f\ (k) - s.f\ (k)$

Let $k = k^*$, be the steady state value of per capita capital. Hence the steady state value of per capita consumption is,

$$c^* = f\ (k^*) - s.f\ (k^*)$$

Again, at steady state we have, $s.f\ (k^*) = (\delta + \mathrm{n})k^*$

$$\Rightarrow c^* = f\ (k^*) - (\delta + \mathrm{n})k^* \qquad (3)$$

Differentiating (3) wrt ‘s’ we get,

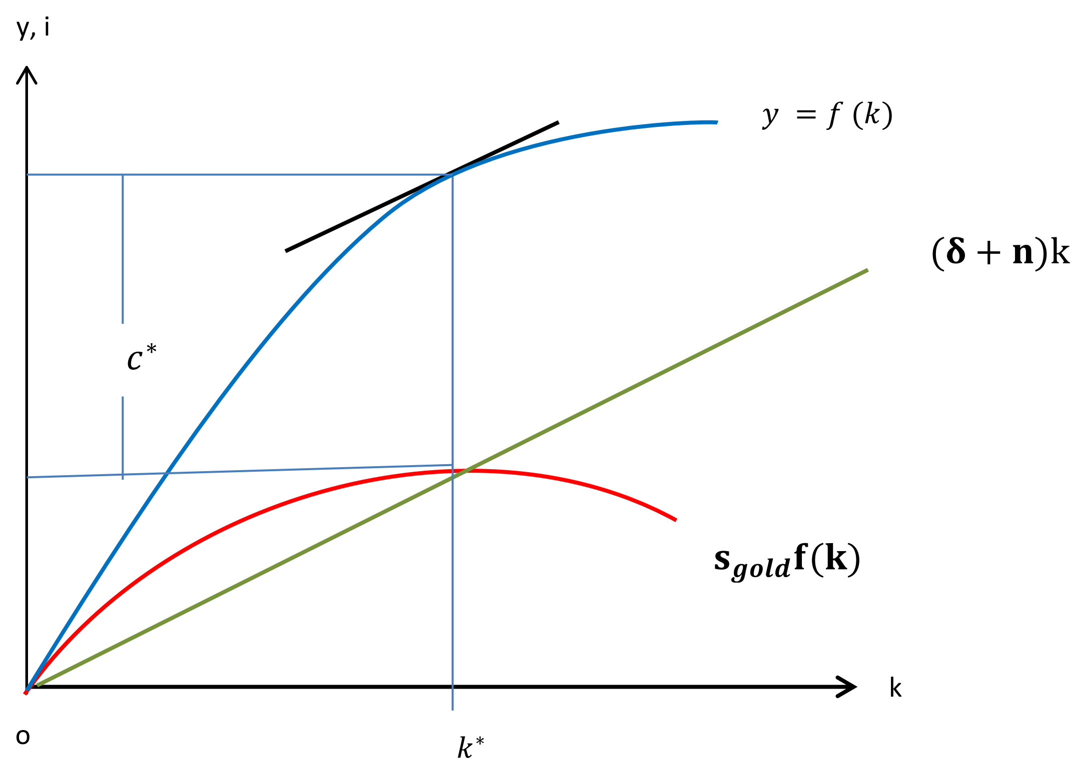
$$\frac{dc^*}{ds} = \frac{dk^*}{ds}[f'(k^*) - (\delta + \mathrm{n})]$$

F.O.C for per capita consumption maximisation requires that,

$$\frac{dc^*}{ds} = \frac{dk^*}{ds}[f'(k^*) - (\delta + \mathrm{n})] = 0$$

$$\Rightarrow f'(k^*) = (\delta + \mathrm{n}),\ \text{since}\ \frac{dk^*}{ds} > 0$$

⇒MP_k = Depriciation Rate + Population Growth Rate



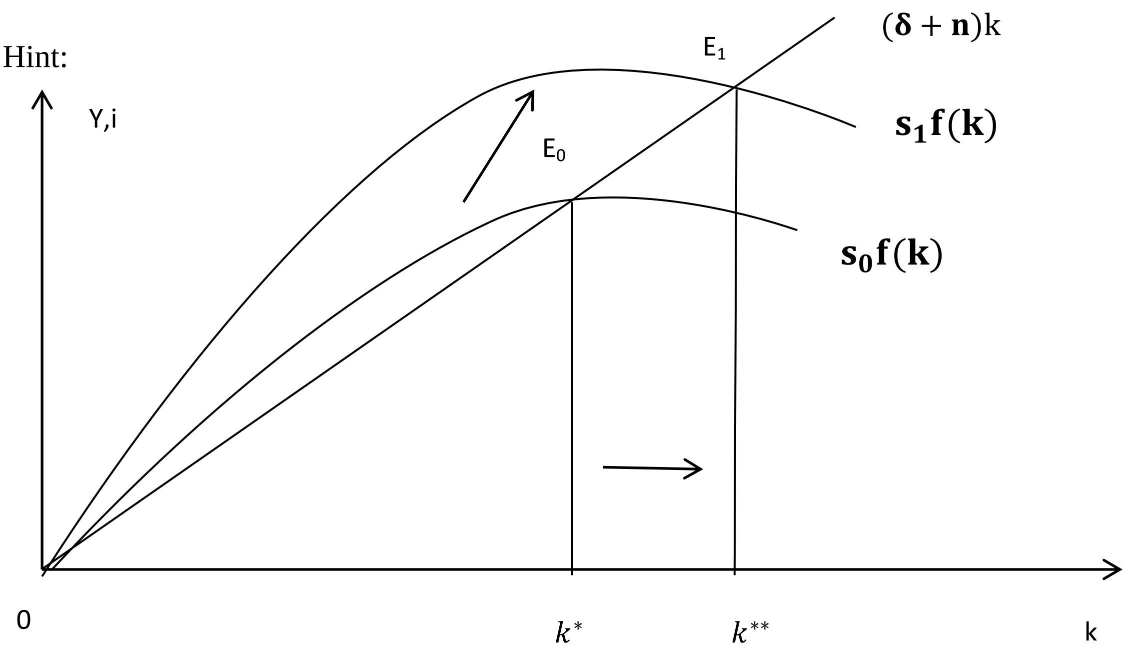
When $s > s_{gold} \Rightarrow f'(k^*) < (\delta + n) \Rightarrow \frac{dc^*}{ds} < 0$

When $s < s_{gold} \Rightarrow f'(k^*) > (\delta + n) \Rightarrow \frac{dc^*}{ds} > 0$

Exercise 1

1.1 Find If s rises then what happens to steady state per capita capital (k^*) and and output growth rate

Hint:



$s_1 > s_0$ ⇒steady state value of per capita capital increases from k^* to k^{**}. In Solow model, the growth rate of output exactly matches the exogeneously given growth rate of labour force. Since there is no change in 'n' therefore output growth rate remains unchanged.

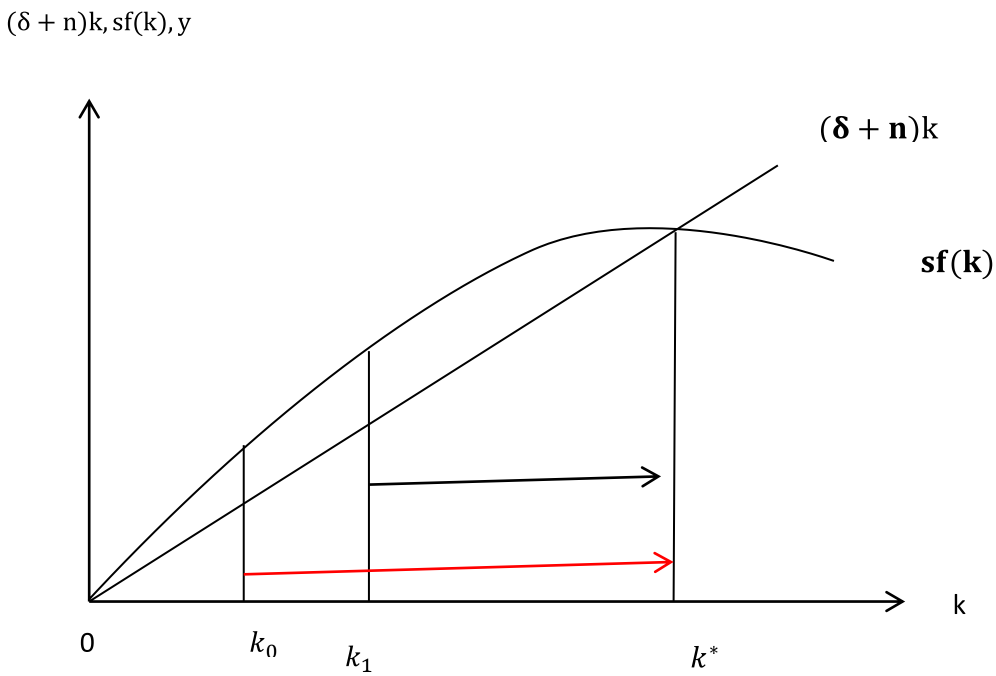
1.2 Find If 'n' increases then what happens to steady state per capita capital (k^*) and and output growth rate.

1.3 Find If 'δ' falls then what happens to steady state per capita capital (k^*) and and output growth rate.

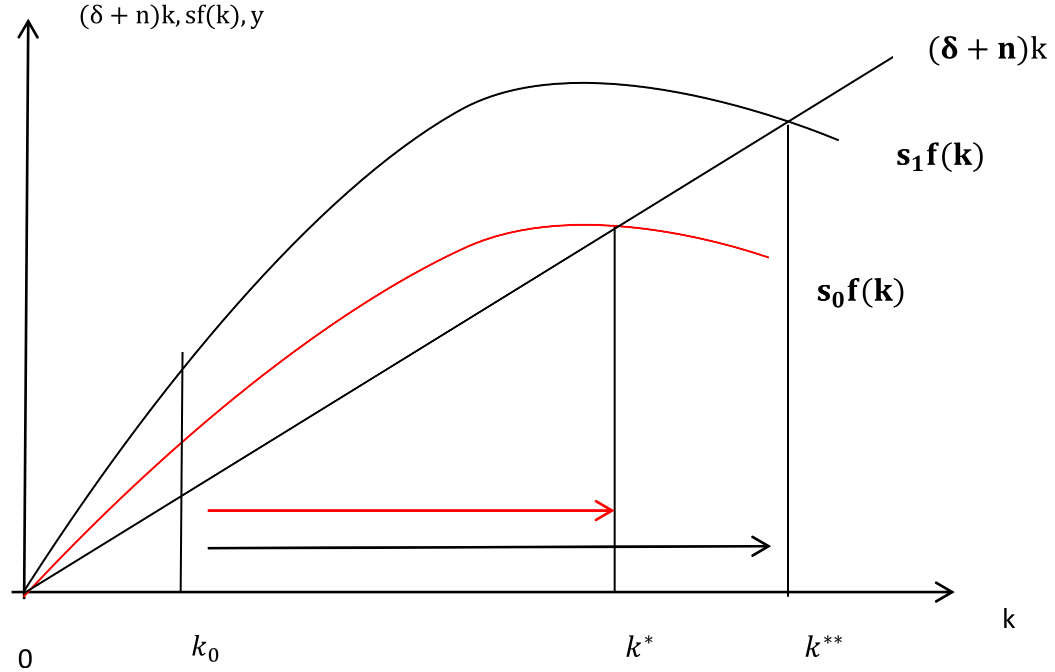
Concept of Convergence

Much research has been devoted to the question whether the economies that start off poor can susequently grow faster than economies that start off rich. If they do then poor economies will tend to catch up with the world's rich countries. This property of catch up is called convergence. Oherwise they will remain poor. The Solow model makes a clear prediction about when the convergence should occur. As per the model whether the two countries will converge depends on why they differ in first place. Now consioder the following cases:

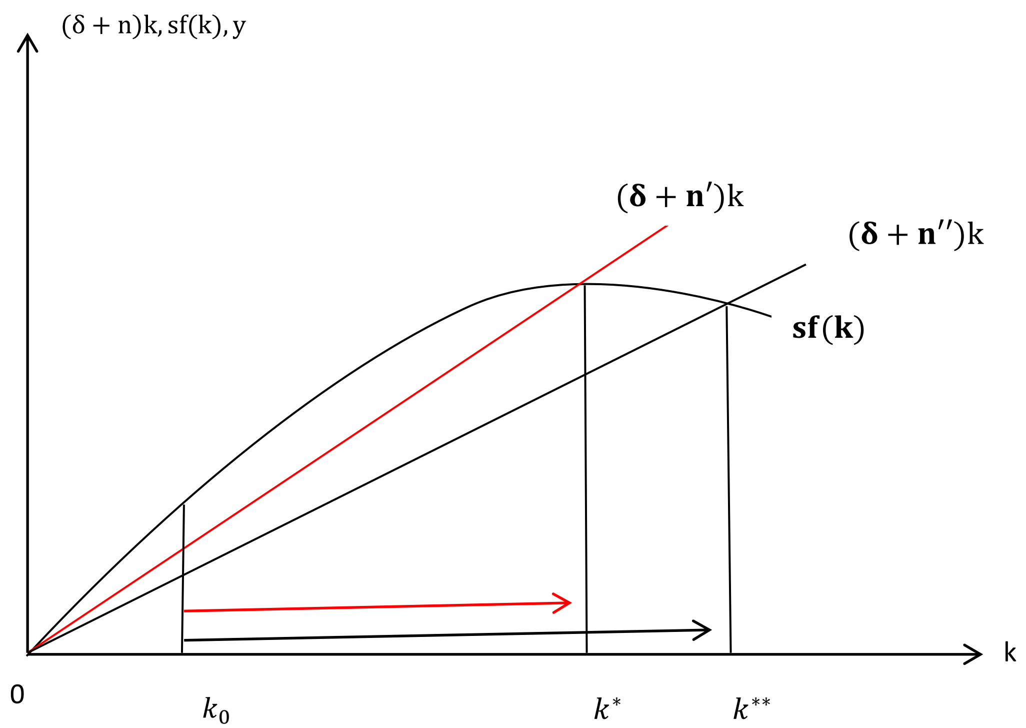
1. Suppose two countries have same population growth rates (n) and savings rate (s). But they differ in initial stock of per capiota capital. Country 1 has k_0 and country has k_1, where $k_0 < k_1$. They are expected to reach the same steady state i.e. the two countries will converge. The poorer country with smaller capital stock will grow more quickly to reach the steady state (see in the graph below).



2. On the other hand, if two economies have different n and s then they are expected to reach their own steady states. That is convergence is not expected.



Countries differ in savings rate, 's'



Countries differ in population growth rate, 'n'

While working with data worldwide it has shown little evidence of convergence. If statistical techniques are used to control for some of the determinants such as savings rate, population growth rates, accumulation of human capital, then the data may exhibit some convergence. In a nutshell, countries are said to have *conditional convergence*: they appear to be converging their own steady states, which in turn are determined by their saving, population growth and human capital.

Structure of Solow Model with Technological Progress

Solow considered labour augmenting techinical progress. The aggregate production function is given by $Y = F(K, EL)$ where E denotes efficiency of labour; EL denotes total number of effective workers; the technologoical progress growth rate is given by 'g', i.e., g= $\frac{1}{E}\frac{dE}{dt} = \frac{\dot{E}}{E}$.

Since Solow assumes constant returns to scale, so $\lambda Y = F(\lambda K, \lambda EL)$ where $\lambda \geq 0$

If we choose $\lambda = 1/EL$ then we can write the above production function as $\frac{Y}{EL} = F(\frac{K}{EL}, 1)$. Now if we define $\frac{K}{EL}$ as k and $\frac{Y}{EL}$ as y then the aggregate production can be rewritten as $\boldsymbol{y} = \boldsymbol{f}\ (\boldsymbol{k})$. This is said to be the per capita production function where y is the per capita output or income and k is the per capita capital. Thus k is the primary variable in solow model.

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$$\Rightarrow \frac{\dot{K}}{K} = \frac{sf(k)}{\frac{K}{EL}} - \delta$$

$\Rightarrow \frac{\dot{K}}{K} = \frac{sf(k)}{k} - \delta$........................ (1)

We know $k = \frac{K}{EL}$

Applying logarithm we have

$$\log k = \log K - logE - \log L$$

Differentiating w.r.t. 't' we get,

$$\frac{1}{k}\frac{dk}{dt} = \frac{1}{K}\frac{dK}{dt} - \frac{1}{E}\frac{dE}{dt} - \frac{1}{L}\frac{dL}{dt}$$

i.e., $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{E}}{E} - \frac{\dot{L}}{L}$

$\Rightarrow \frac{\dot{k}}{k} = \frac{sf(k)}{k} - \delta - g - n$ (from 1)

$\Rightarrow \dot{\mathbf{k}} = \mathbf{sf}(\mathbf{k}) - (\boldsymbol{\delta} + \mathbf{n} + \mathbf{g})k$ .... (2)

<u>(2) is the basic differential equation of solow model.</u>

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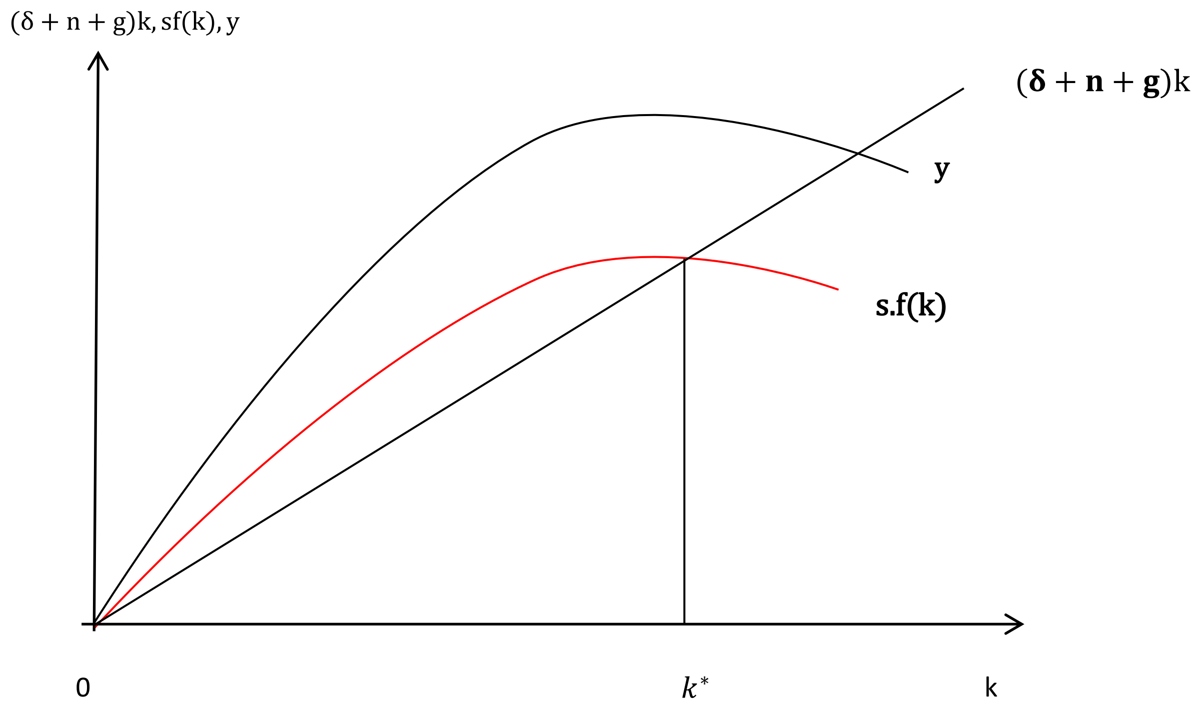
Diagrammatic representation of steady-state



Explanation same as before.

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Differentiating (3) wrt ‘s’ we get,

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F.O.C for per capita consumption maximisation requires that,

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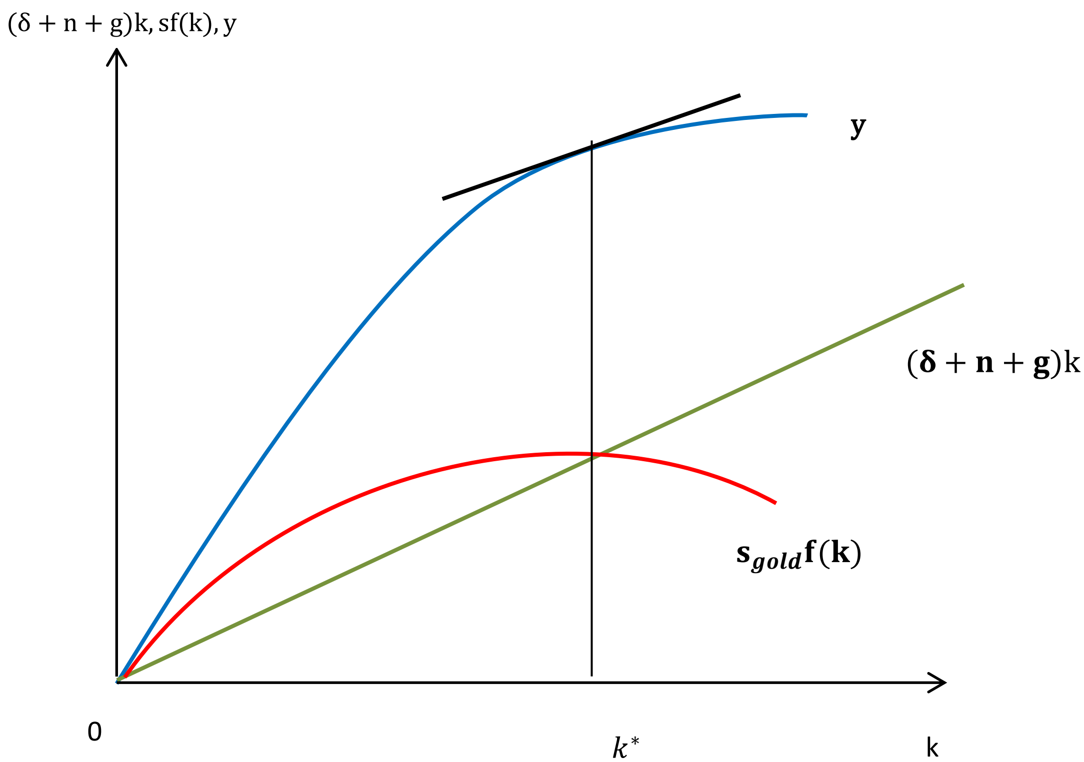
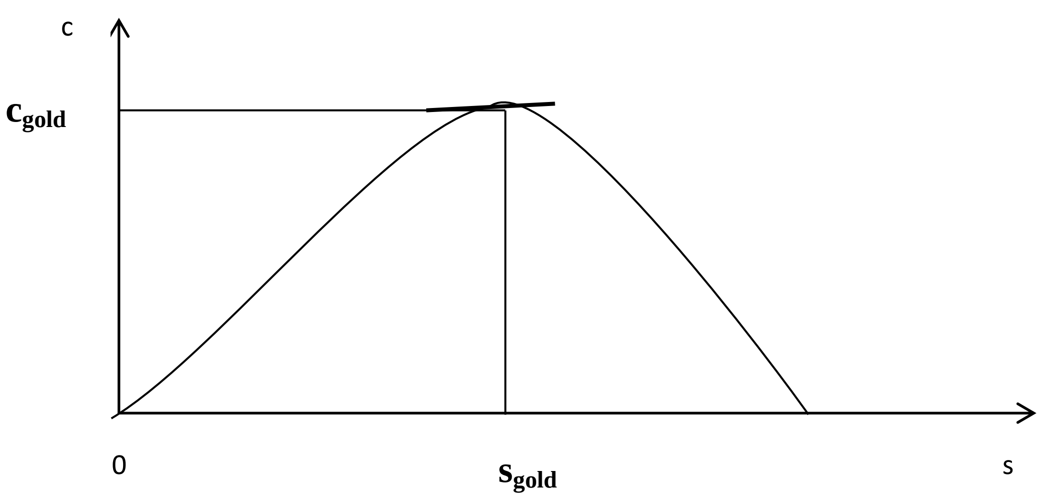
$$\Rightarrow f'(k^*) = (\delta + n + g), \text{ since } \frac{dk^*}{ds} > 0$$

⇒MP_k = Depriciation Rate + Population Growth Rate+ growth rate of technological progress



Explanation same as before. The relationship between golden rule saving rate and golden rule per capita consumption can be depicted by the following diagram.



Solow Residual is defined as the rising productivity as rising output with constant K, L as inputs. It is *residual* because it is that part of growth which is not accounted for by measures of capital accumulation or increased labour input.