Neutrino Oscillation in non-uniform matter

$$i\frac{d}{dx}\nu^{(f)} = \left[E + \frac{m_1^2+m_2^2}{4E} - \frac{1}{\sqrt{2}}G_F n_n + \frac{1}{2E}\tilde{M}^2\right]\nu^{(f)}$$

unit matrix term is omitted as they do not affect the ~~probabilite~~ probability

Now, $\nu^{(f)} = \tilde{U}\tilde{\nu}^{(p)}$

in the equation $i\frac{d\nu^{(f)}}{dx} = \frac{1}{2E}\tilde{M}^2\nu^{(f)}$

$$\Rightarrow \quad i\frac{d}{dx}\left[\tilde{U}\tilde{\nu}^{(p)}\right] = \frac{1}{2E}\tilde{M}^2\tilde{U}\tilde{\nu}^{(p)}$$

For non-uniform matter $\tilde{\theta}$ and so $\tilde{U}$ are different at different positions.

$$\Rightarrow \quad i\tilde{U}\frac{d}{dx}\tilde{\nu}^{(p)} + i\left(\frac{d\tilde{U}}{dx}\right)\tilde{\nu}^{(p)} = \frac{1}{2E}\tilde{M}^2\tilde{U}\tilde{\nu}^{(p)}$$

$$\Rightarrow \quad i\frac{d\tilde{\nu}^{(p)}}{dx} = \left[\frac{1}{2E}\tilde{U}^{\dagger}\tilde{M}^2\tilde{U} - i\tilde{U}^{\dagger}\frac{d\tilde{U}}{dx}\right]\tilde{\nu}^{(p)}$$

Now, $$\frac{1}{2E}\tilde{U}^{\dagger}\tilde{M}^2\tilde{U}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\end{pmatrix} = \begin{pmatrix}\frac{\tilde{m}_1^2}{2E} & 0\\ 0 & \frac{\tilde{m}_2^2}{2E}\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\end{pmatrix}$$

and $\cancel{\Rightarrow}$ $$\tilde{U}^{\dagger}\frac{d\tilde{U}}{dx} = \begin{pmatrix}0 & -\frac{d\tilde{\theta}}{dx}\\ \frac{d\tilde{\theta}}{dx} & 0\end{pmatrix}$$

$$\therefore \quad i\frac{d\tilde{\nu}^{(p)}}{dx} = i\frac{d}{dx}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\end{pmatrix} = \begin{pmatrix}\frac{\tilde{m}_1^2}{2E} & i\frac{d\tilde{\theta}}{dx}\\ -i\frac{d\tilde{\theta}}{dx} & \frac{\tilde{m}_2^2}{2E}\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\end{pmatrix}$$

For $\frac{d\tilde{\theta}}{dx} = 0$ $\quad \tilde{\nu}_1$ & $\tilde{\nu}_2 \rightarrow$ stationary eigen states

The solutions are of two types

(1) Adiabatic solution $\Rightarrow \frac{d\tilde{\theta}}{dx}$ is small

(2) Non-adiabatic solution $\Rightarrow \frac{d\tilde{\theta}}{dx}$ is not small enough

Adiabatic parameter.

Adiabatic solution is obtained by assuming $\tilde{\theta}$ is slowly varying parameter

Adiabatic condition : $\left|\frac{d\tilde{\theta}}{dx}\right| \ll \left|\frac{\tilde{m}_2^2 - \tilde{m}_1^2}{2E}\right|$

Adiabatic parameter : $\gamma = \left|\frac{(\tilde{m}_2^2 - \tilde{m}_1^2)/2E}{\frac{d\tilde{\theta}}{dx}}\right|$

$$\tan 2\tilde{\theta} = \frac{\Delta \sin 2\theta}{\Delta \cos 2\theta - A} \qquad \sin 2\tilde{\theta} = \frac{\Delta \sin 2\theta}{\left[(\Delta \cos 2\theta - A)^2 + \Delta^2 \sin^2 2\theta\right]^{1/2}}$$

Again,

$$\tilde{m}_2^2 = \frac{1}{2}\left[(m_1^2 + m_2^2 + A) + \sqrt{(\Delta \cos 2\theta - A)^2 + \Delta^2 \sin^2 2\theta}\right]$$

$$\tilde{m}_2^2 = \frac{1}{2}\left[(m_1^2 + m_2^2 + A) - \sqrt{(\Delta \cos 2\theta - A)^2 + \Delta^2 \sin^2 2\theta}\right]$$

$$\boxed{\tilde{m}_2^2 - \tilde{m}_1^2 = \sqrt{(\Delta \cos 2\theta - A)^2 + \Delta^2 \sin^2 2\theta} = \frac{\Delta \sin 2\theta}{\sin 2\tilde{\theta}}}$$

$$A = 2\sqrt{2}\, G_F E n_e$$

$$\Rightarrow \frac{dA}{dx} = 2\sqrt{2}\, G_F E \frac{dn_e}{dx}$$

Now $\tan 2\tilde{\theta} = \dfrac{\Delta \sin 2\theta}{\Delta \cos 2\theta - A}$

Differentiate both sides w.r.t. x

$$2 \frac{d\tilde{\theta}}{dx} \sec^2 2\tilde{\theta} = -\frac{\Delta \sin 2\theta}{(\Delta \cos 2\theta - A)^2} \frac{dA}{dx}$$

$$\Rightarrow \frac{d\tilde{\theta}}{dx} = -\sqrt{2}\, G_F E \frac{\sin^2 2\tilde{\theta}}{\Delta \sin 2\theta} \frac{dn_e}{dx}$$

[Using $\sec^2 2\tilde{\theta} = \dfrac{(\Delta \cos 2\theta - A)^2 + \Delta^2 \sin^2 2\theta}{(\Delta \cos 2\theta - A)^2}$

and $\sqrt{(\Delta \cos 2\theta - A)^2 + \Delta^2 \sin^2 2\theta} = \dfrac{\Delta \sin 2\theta}{\sin 2\tilde{\theta}}$]

Now, $\dfrac{1}{\left|\dfrac{d\tilde{\theta}}{dx}\right|} = \left(\dfrac{\Delta}{E}\right) \dfrac{1}{2\sqrt{2}\, G_F} \dfrac{\sin 2\theta}{\sin^2 2\tilde{\theta}} \dfrac{1}{\left|\dfrac{dn_e}{dx}\right|}$

$$\therefore\ \gamma(x) = \left| \frac{\tilde{m}_2^2 - \tilde{m}_1^2 / 2E}{\frac{d\tilde{\theta}}{dx}} \right| = \frac{(\Delta/E)^2}{2\sqrt{2}\, G_F} \frac{\sin^2 2\theta}{\sin^3 2\tilde{\theta}} \frac{1}{\left|\frac{dn_e}{dx}\right|}$$

At resonance

$$\left[\text{i.e at } \Delta \cos 2\theta = A_R \right]$$

$$\gamma = \gamma_R$$

$$\Rightarrow \quad \sin 2\theta \big|_R = 1$$

and $\quad A_R = \Delta \cos 2\theta = 2\sqrt{2}\, G_F E\, n_R$

$$\Rightarrow \quad n_R = \frac{\Delta \cos 2\theta}{2\sqrt{2}\, G_F E}$$

Now $\quad \frac{d}{dx}\left(\ln n_e\right) = \frac{1}{n_e} \frac{dn_e}{dx}$

$$\Rightarrow \quad \frac{1}{\left|\frac{dn_e}{dx}\right|} = \frac{1}{n_e \left|\frac{d}{dx}\left(\ln n_e\right)\right|}$$

$$\Rightarrow \quad \frac{1}{\left|\frac{dn_e}{dx}\right|_R} = \frac{2\sqrt{2}\, G_F E}{\Delta \cos 2\theta} \frac{1}{\left|\frac{d}{dx} \ln n_e\right|_R}$$

At resonance

$$\gamma_R = \left(\frac{\Delta}{E}\right) \frac{\sin^2 2\theta}{\cos 2\theta} \frac{1}{\left|\frac{d}{dx} \ln n_e\right|_R}$$

As the maximum mixing ~~occuring~~ occur near resonance therefore adiabacity occurs if

$$\gamma_R \gg 1$$

$\Rightarrow$ propogation is adiabatic

Adiabatic solution

[$\frac{d\tilde{\theta}}{dx}$ is small]

Particular case : $A = A_0$ (Point where neutrino is created, at solar core)

If $A_0 \to \infty \Rightarrow n_e \to \infty$

$$\tan 2\tilde{\theta}_0 = \frac{\Delta \sin 2\theta}{\Delta \cos 2\theta - A_0} \Rightarrow \tilde{\theta}_0 \to \frac{\pi}{2}$$

[$\tan 2\tilde{\theta}_0$ is negative]

$$\tilde{\nu}_1 = \nu_e \cos\tilde{\theta}_0 - \nu_\mu \sin\tilde{\theta}_0$$
$$\tilde{\nu}_2 = \nu_e \sin\tilde{\theta}_0 + \nu_\mu \cos\tilde{\theta}_0$$

$$\tilde{\theta}_0 \to \frac{\pi}{2} \Rightarrow \nu_e \simeq \tilde{\nu}_2$$

→ In that case ν_e beam is created at the core of the sun

→ ν_e beam traverses outside the sun as $\tilde{\nu}_2$

→ Outside the sun $A = 0$.

$$\Rightarrow \nu_2 = \nu_e \sin\theta + \nu_\mu \cos\theta$$

therefore, probability of finding ν_e

$$\to |\langle \nu_e | \nu_2 \rangle|^2 = \sin^2\theta$$

$$\Rightarrow P^{(ad)}_{\nu_e \nu_e} = \sin^2\theta \quad \text{for } A_0 \to \infty$$

General Case

$$\nu_e = \tilde{\nu}_1 \cos\tilde{\theta}_0 + \tilde{\nu}_2 \sin\tilde{\theta}_0$$

⇒ Probability that neutrino is produced as $\tilde{\nu}_1$ is $\cos^2\tilde{\theta}_0$

→ Neutrino travels as $\tilde{\nu}_1$ outside the sun where $A = 0$

⇒ $\tilde{\nu}_1$ becomes $\nu_1 = \nu_e \cos\theta - \nu_\mu \sin\theta$

→ Outside the sun ν_1 can be detected as ν_e with probability $\cos^2\theta$

⇒ Neutrino ν_e may be produced as $\tilde{\nu}_1$ with probability ~~$\sin^2\tilde{\theta}_0$ and can be~~ $\cos^2\tilde{\theta}_0$ and can be detected as ν_e outside the sun with probability ~~$\sin^2\theta$~~ $\cos^2\theta$

⇒ The total probability = ~~$\cos^2\tilde{\theta}_0 \sin^2\theta$~~ $\cos^2\tilde{\theta}_0 \cos^2\theta$

There is another channel

The neutrino ν_e may be produced as $\tilde{\nu}_2$ with probability $\sin^2\tilde{\theta}_0$ and can be detected as ν_e outside the sun with probability $\sin^2\theta$

⇒ The total probability = $\sin^2\tilde{\theta}_0 \sin^2\theta$

$$\Rightarrow P^{(ad)}_{\nu_e\nu_e} = \cos^2\tilde{\theta}_0 \cos^2\theta + \sin^2\tilde{\theta}_0 \sin^2\theta = \cancel{\frac{1}{2}\left[1 + \cos^2 2\tilde{\theta}_0 \cos 2\theta\right]} = \frac{1}{2}\left[1 + \cos 2\tilde{\theta}_0 \cos 2\theta\right]$$

$$P^{(ad)}_{\nu_e\nu_\mu} = 1 - P^{(ad)}_{\nu_e\nu_e} = \cancel{\frac{1}{2}\left[1 - \cos^2 2\tilde{\theta}_0 \cos 2\theta\right]} = \frac{1}{2}\left[1 - \cos 2\tilde{\theta}_0 \cos 2\theta\right]$$

More general expression

$$P^{(ad)}_{\nu_e\nu_e}(x) = \frac{1}{2}\left[1 + \cos 2\tilde{\theta}_0 \cos 2\tilde{\theta} + \sin 2\tilde{\theta}_0 \sin 2\tilde{\theta} \cos\left\{\int_0^x (E_2 - E_1)\, dx'\right\}\right]$$