Line Integrals

Let Γ denotes a portion of a regular curve $\vec{r} = \vec{r}(s)$ and a vector $\vec{F}$ be defined at all points of Γ. Then the line integral of the vector function $\vec{F}$ taken over Γ is defined as

$$\int_{\Gamma} \vec{F}\cdot d\vec{r} = \int_{\Gamma} \vec{F}\cdot\vec{t}\, ds = \int_{s_A}^{s_B} F_t\, ds \qquad \Big| \; \Gamma : s_A \le s \le s_B$$

* The line integral of a vector function is really the ordinary Riemann integral along the curve of the tangential component of the vector.

In case Γ is a closed curve it is customary to denote the line integral by $\oint_{\Gamma} \vec{F}\cdot d\vec{r}$

e.g. – $\vec{F} = x^2\hat{i} + y^3\hat{j}$ and Γ : a portion of $y = x^2$ in the xy-plane from $A(0,0)$ to $B(1,1)$

Ans: $y = x^2 \Rightarrow x = t$ and $y = t^2$ (parametric form)

$A(0,0) \to B(1,1) \Rightarrow t = 0 \to t = 1$ $\quad\Big|\; \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\int_{\Gamma} \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} \left(x^2\frac{dx}{dt} + y^3\frac{dy}{dt} \right) dt = \int_0^1 \left(t^2 + t^6 \cdot 2t^2 \right) dt$$

$$= \frac{7}{12}$$

Note – $\vec{F} = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}$

and $\vec{r}(s) = x(s)\,\hat{i} + y(s)\,\hat{j} + z(s)\,\hat{k}$

$$\int_{\Gamma} \vec{F}\cdot d\vec{r} = \int_{\Gamma} \left(F_1\,dx + F_2\,dy + F_3\,dz \right) = \int_{s_A}^{s_B} \left(F_1\frac{dx}{ds} + F_2\frac{dy}{ds} + F_3\frac{dz}{ds} \right) ds$$

Simply-connected region

A region R is said to be simply-connected if every closed curve in the ~~of~~ region can be shrunk continuously to a point in the region.

e.g. - interior of a sphere

Circulation.

Let $\vec{F}$ be a vector and Γ be a closed curve. The value of the line integral $\oint_\Gamma \vec{F}\cdot d\vec{r}$ around the closed curve Γ is called the circulation.

Irrotational vector field.

The vector field $\vec{F}$ defined over a region R is said to be irrotational if the circulation $\oint_\Gamma \vec{F}\cdot d\vec{r}$ around all closed curves in R is zero.

Th - A necessary and sufficient condition that a continuous vector field $\vec{F}$ be irrotational in a simply connected region R is that a single valued scalar function ϕ exists for which

~~$\vec{F}$~~ $\vec{F} = \vec{\nabla}\phi = \text{grad}\,\phi$

Proof of sufficient condition:

Let $\exists$ $\phi(x,y,z)$ s.t. $\vec{\nabla}\phi = \vec{F}$

$$\oint_\Gamma \vec{F}\cdot d\vec{r} = \oint_\Gamma (\vec{\nabla}\phi)\cdot\vec{t}\,ds \qquad [d\vec{r} = \vec{t}\,ds]$$

$$= \oint_\Gamma \frac{d\phi}{ds}\,ds \quad [\text{directional derivative along } \vec{t}\,]$$

$$= \oint_\Gamma d\phi = [\phi]_\Gamma = 0 \quad [\because \Gamma \text{ is closed curve}]$$

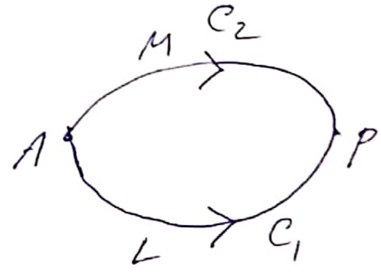
$\because \Gamma$ is arbitrary $\Rightarrow \vec{F}$ is irrotational

Conservative vector field

Let $\vec{F}$ be a vector field defined over a region R. The vector field $\vec{F}$ is said to be ~~conservative~~ conservative if $\exists$ a scalar $\phi(x,y,z)$ s.t. $\vec{F} = \vec{\nabla}\phi = \text{grad}\,\phi$ is satisfied. $\boxed{\phi \to \text{Potential function}}$

If $\vec{F}$ is conservative then the line integral $\int_{\Gamma} \vec{F}\cdot d\vec{r}$ over the curve ~~is~~ joining any two given points is independent of the path.

Proof: $\vec{F} = \vec{\nabla}\phi \Rightarrow \vec{F}$ is irrotational



Let Γ be a curve from A to P

C_1 and C_2 ~~be~~ ~~be~~ are two paths from A to P s.t. $C_1 : AMP$ and $C_2 : ALP$

Let $C : ALPMA$ be a closed curve

then $C \equiv C_1 - C_2$

$\because \vec{F}$ is irrotational $\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = 0$

$$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = 0$$

$$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} = 0$$

$$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

$\Rightarrow \int \vec{F}\cdot d\vec{r}$ depends only on A and P.

Alternative definition of conservative field

If the line integral of a vector field $\vec{F}$ depends on $\vec{F}$, and on the end points A, B but not on the nature of the path joining them A and B, then the vector field $\vec{F}$ is said to be a conservative field.

Under what condition a vector field $\vec{F}$ is conservative?

If $\vec{F}$ be conservative, then $\exists$ a $\phi(x,y,z)$

s.t. $\vec{F} = \vec{\nabla}\phi$

Now $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times(\vec{\nabla}\phi) = 0$ [Proof left as exercise]

Thus $\vec{\nabla}\times\vec{F} = 0$ is the condition of a conservative field

Ex - Prove that $f\,dx + g\,dy + h\,dz$ is an exact differential iff the vector function $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$ is conservative.

Proof: $\vec{F} = \vec{\nabla}\phi \iff f\hat{i} + g\hat{j} + h\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

i.e. iff $f = \frac{\partial\phi}{\partial x}$, $g = \frac{\partial\phi}{\partial y}$, $h = \frac{\partial\phi}{\partial z}$

i.e. iff $f\,dx + g\,dy + h\,dz = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$

i.e. $\vec{F} = \vec{\nabla}\phi$ iff $f\,dx + g\,dy + h\,dz \to$ exact differential ϕ

Note - If $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$, then $f\,dx + g\,dy + h\,dz$ is an exact differential iff $\vec{\nabla}\times\vec{F} = 0$

or Curl $\vec{F}$ = Rot $\vec{F} = 0$

Work done by a Force

If $\vec{F}$ be the force acting on a particle moving along a curve Γ then the integral

$$\int_{\Gamma} \vec{F}\cdot d\vec{r} = \int_{\Gamma} (F_1 dx + F_2 dy + F_3 dz)$$

represents the work done by the force $\vec{F}$ in describing the curve Γ.

ex-1 Find the total work done in moving a ~~particle~~ particle in a force field given by

$$\vec{F} = (2x - y + z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$$

along a circle $C: x^2 + y^2 = 9,\ z = 0$

Ans. Work done $= \oint_C \vec{F}\cdot d\vec{r}$

$$= \oint_C (2x - y + z)dx + (x + y - z)dy + (3x - 2y - 5z)dz$$

$$= \oint_C \{(2x - y)\,dx + (x + y - z)\,dy\}$$

$$[\ z = 0 \Rightarrow dz = 0\ ]$$

$$= \int_0^{2\pi} \{(6\cos t - 3\sin t)(-3\sin t\,dt) + (3\cos t + 3\sin t)(3\cos t\,dt)\}$$

$$[\ C: x = 3\cos t,\ y = 3\sin t,\ z = 0 \quad 0 \le t \le 2\pi\ ]$$

$$= 18\pi \text{ (units of work)}$$

Ex-2

Show that the work done on an object by a force $\vec{F}$ during a displacement A to B is equal to the change in K.E. $\frac{1}{2}mv^2$ of the object.

Ans. Work done by $\vec{F}$

$$= \int_A^B \vec{F} \cdot d\vec{r}$$

$$\vec{F} = m\frac{d\vec{v}}{dt} \text{ (Newton's law)}$$

$$d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{v}\,dt$$

$$= \int_A^B m\frac{d\vec{v}}{dt} \cdot \vec{v}\,dt$$

$$= \frac{1}{2}\int_A^B m\frac{d}{dt}(\vec{v} \cdot \vec{v})\,dt$$

$$= \frac{1}{2}m\left(v_B^2 - v_A^2\right) = \text{change in K.E.}$$

Note 1 -

If $\vec{F} = \vec{\nabla}\phi$ and path of the integration is a closed curve Γ, then

$$\text{Work done} = \oint_\Gamma \vec{F} \cdot d\vec{r} = \oint_\Gamma (\vec{\nabla}\phi) \cdot \vec{t}\,ds$$

$$= \oint_\Gamma d\phi = 0$$

$\Rightarrow$ change in K.E $= 0$ and then

$\Rightarrow$ the energy is conserved and the conservative field of force.

$\vec{F}$ is called the conservative field of force.

Note-2 -

At any position P an object subject to a conservative force $\vec{F}$ is said to possess a Potential Energy relative to fixed point A.

$$P.E = \int_A^P \vec{F} \cdot d\vec{r}$$