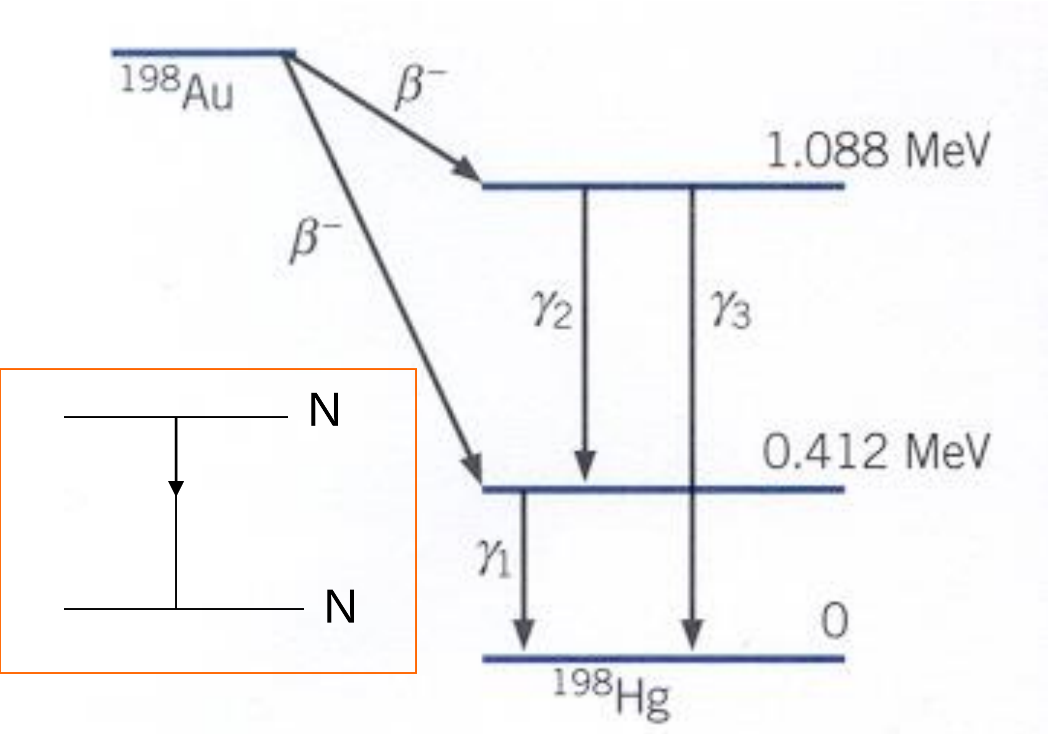
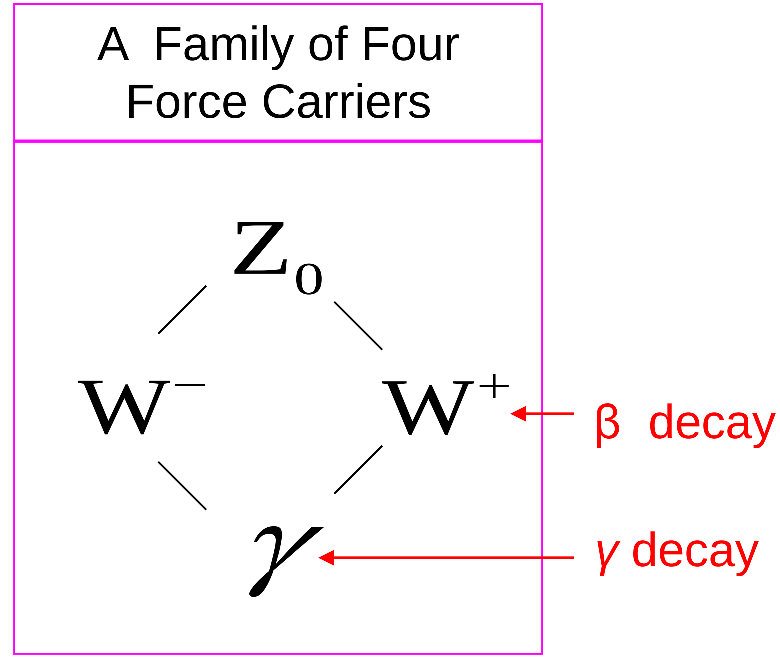
Beta Decay
Gamma Decay

Acknowledgement: The University of Hong Kong

Gamma and Beta decays are similar





Unlike α decay, β and γ decays are closely related (e.g. like cousins).

- They often occur together as in the typical decay scheme (i.e. ^{198}Au)
- They just involved changes in nucleon states (p $\longrightarrow$ n, n $\longrightarrow$ p, p $\longrightarrow$ p)
- They involve the same basic force (γ, $W^{\pm}$) carrier but in different state
- But β decays are generally much slower (~100,000) than γ decays (produced by **EM force**) because the Ws are heavy particles (which makes force **weaker**)

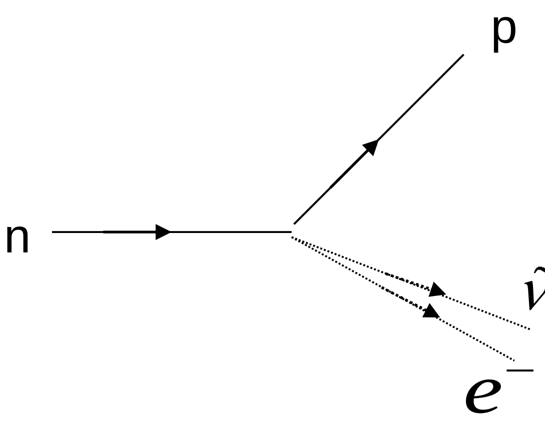
Gamma and Beta decays are very similar

Decay	Name of process	Interaction	Out Channel
$p_i \longrightarrow p_f + \gamma$	Gamma Decay	EM	Nucleon + **Zero** Leptons
$p_i + e^-_{atom} \longrightarrow p_f + e^-_{free}$	Internal Conversion	EM	Nucleon + **One** Lepton
$p + e^-_{atom} \longrightarrow n + \nu_e$	Electron Capture	weak	
$p_i \longrightarrow p_f + e^+ + e^-$	Pair Internal Conversion	EM	Nucleon + **Two** Leptons
$p \longrightarrow n + e^+ + \nu_e$	β^+ Decay	weak	
$n \longrightarrow p + e^- + \tilde{\nu}_e$	β^- Decay	weak	

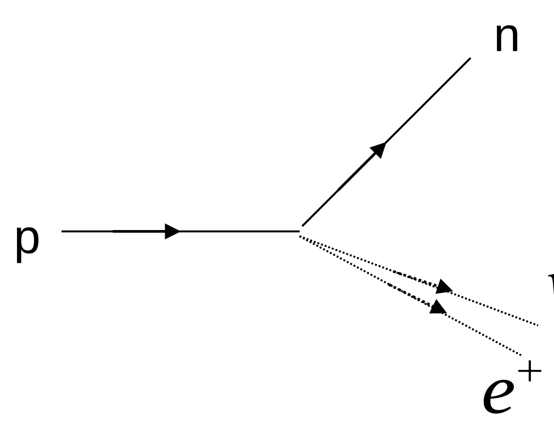
Feynman Diagrams - Similarity

OUT CHANNEL ---- One nucleon + 2 leptons

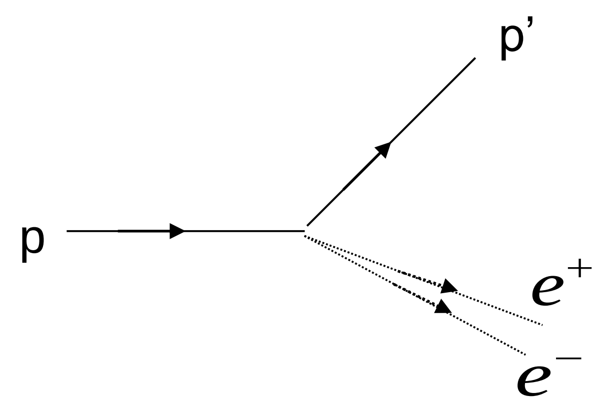
TIME



BETA MINUS DECAY



BETA PLUS DECAY



PAIR INTERNAL CONVERSION

All these decay types are similar in structure

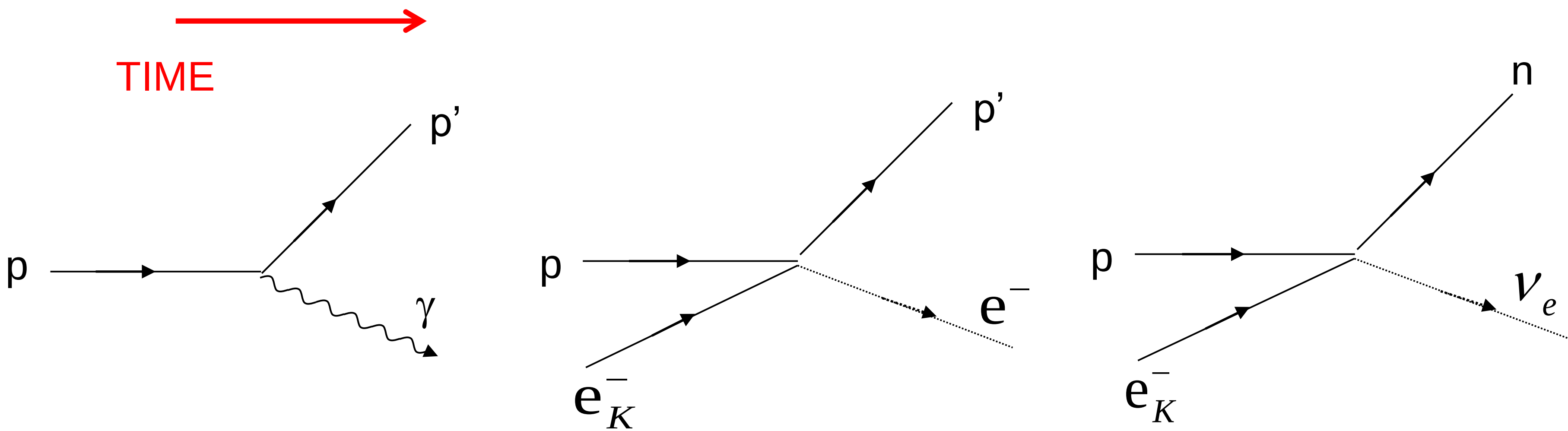
They all have a **4 point vertex**

They all have **3 particles in the final state**

The fact that the Q of the decay is shared between 3 particles means that the outgoing observed particle [ie. electron or positron] has a **spectrum of energies** in the range (0 to Q).

Feynman Diagrams - Similarity

OUT CHANNEL ---- One nucleon + 1 lepton



GAMMA DECAY

[Mono-energetic photons]

INTERNAL CONVERSION

[Mono-energetic electrons]

ELECTRON CAPTURE

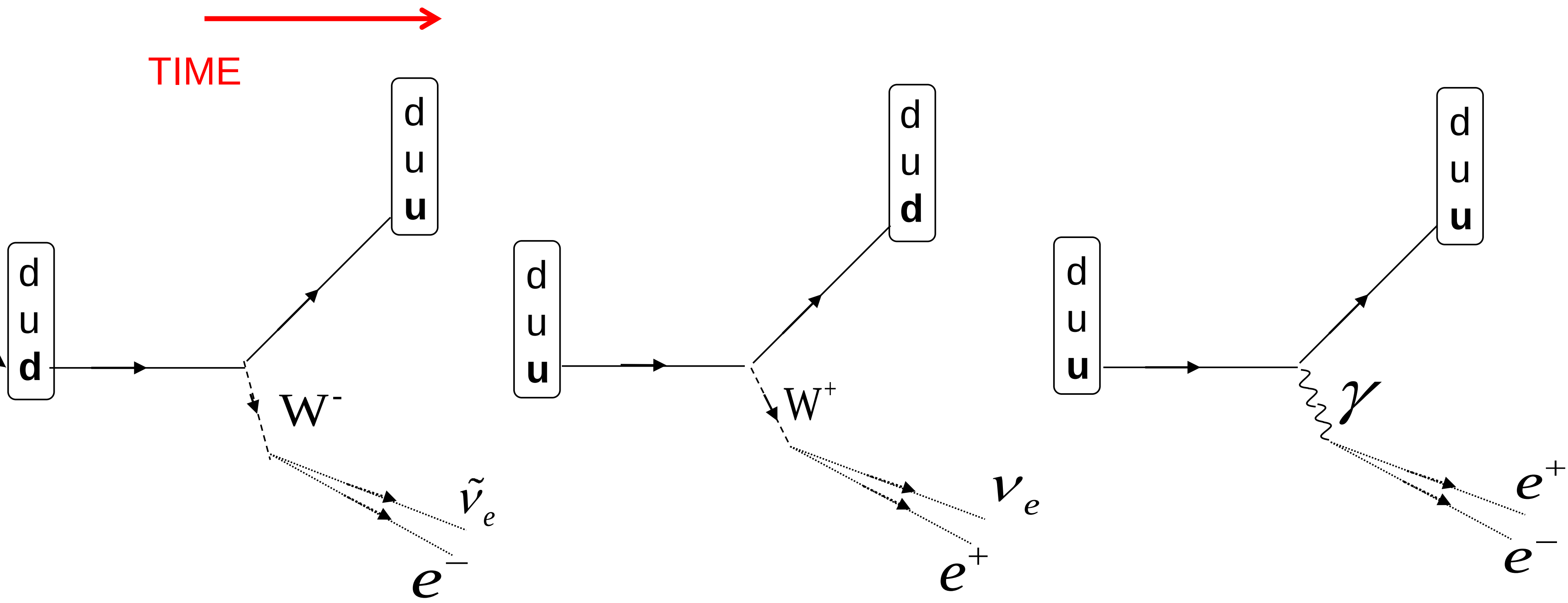
[Mono-energetic neutrinos]

All these decays have only **two particles in their output** state.

The Q of the decay is shared between only 2 particles

Conservation of Energy: The emitted particle (γ , e^-, ν_e) is **monoenergetic**.

Quark level Feynman Diagrams - Similarity



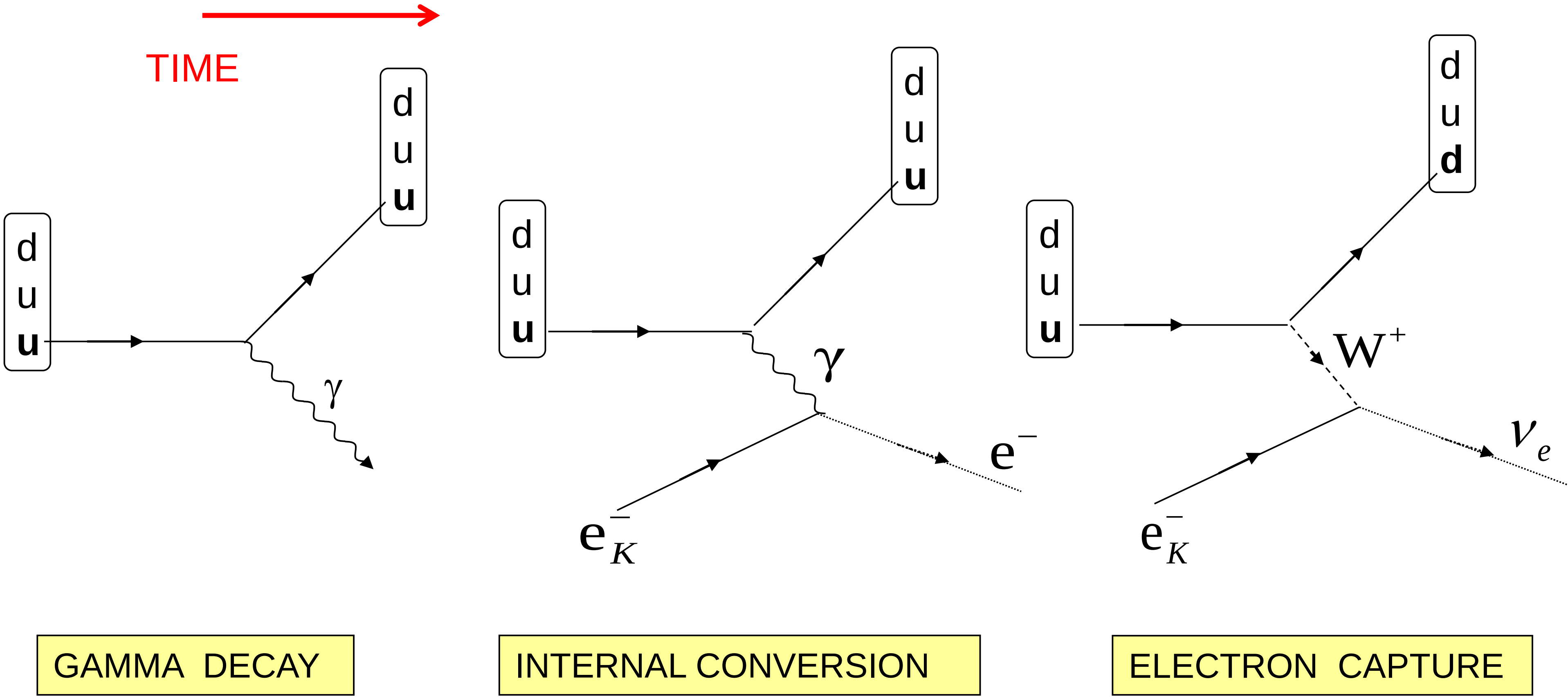
BETA MINUS DECAY | BETA PLUS DECAY | PAIR INTERNAL CONVERSION

The **proton is made of 3 quarks** – uud (up, up, down)

The **neutron is made also of 3 quarks** - udd (up, down, down)

We see the very close similarity of pattern between reactions through W and γ particles. NOTE: only vertices of 3 particles are now seen (makes sense)

Quark level Feynman Diagrams - Similarity

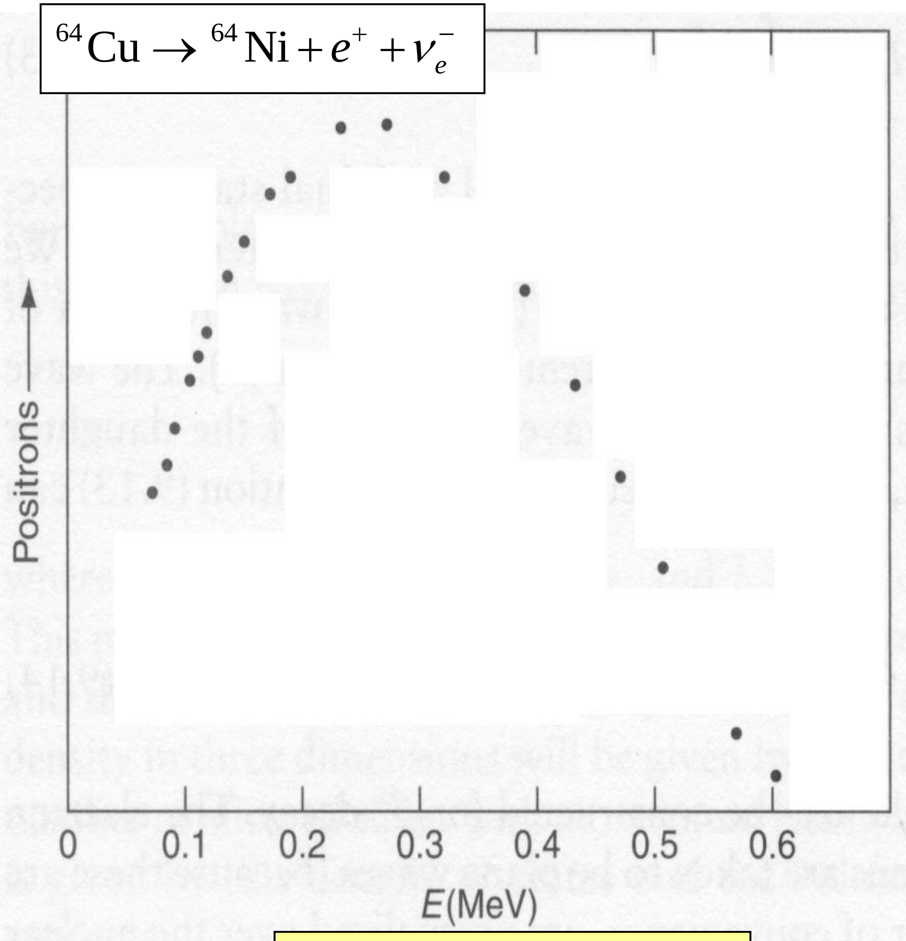


Again we see that there are **ONLY 3 PARTICLE – VERTICES**

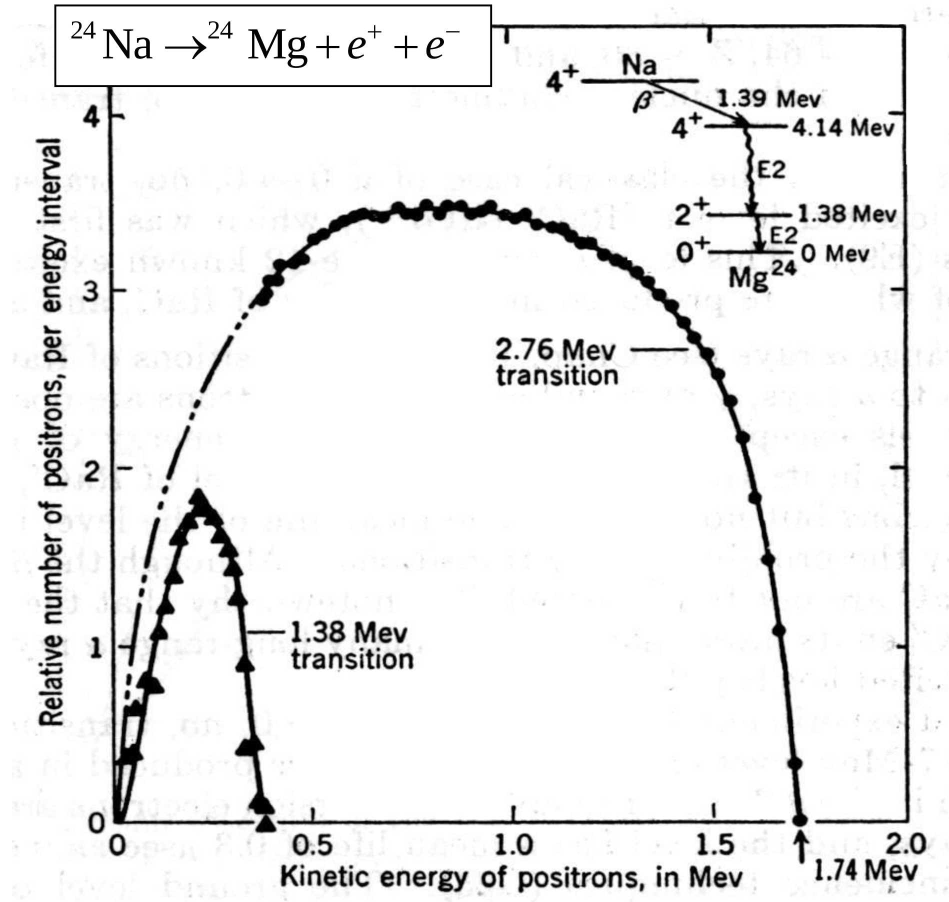
We see the similarity of the decays are propogated through the **intermedicate "Force" particles (W and γ).**

Remember in INTERNAL CONV. And ELECTRON CAPTURE the electron comes from the **core electron orbitals of THE ATOM**.

Beta and Gamma similarities



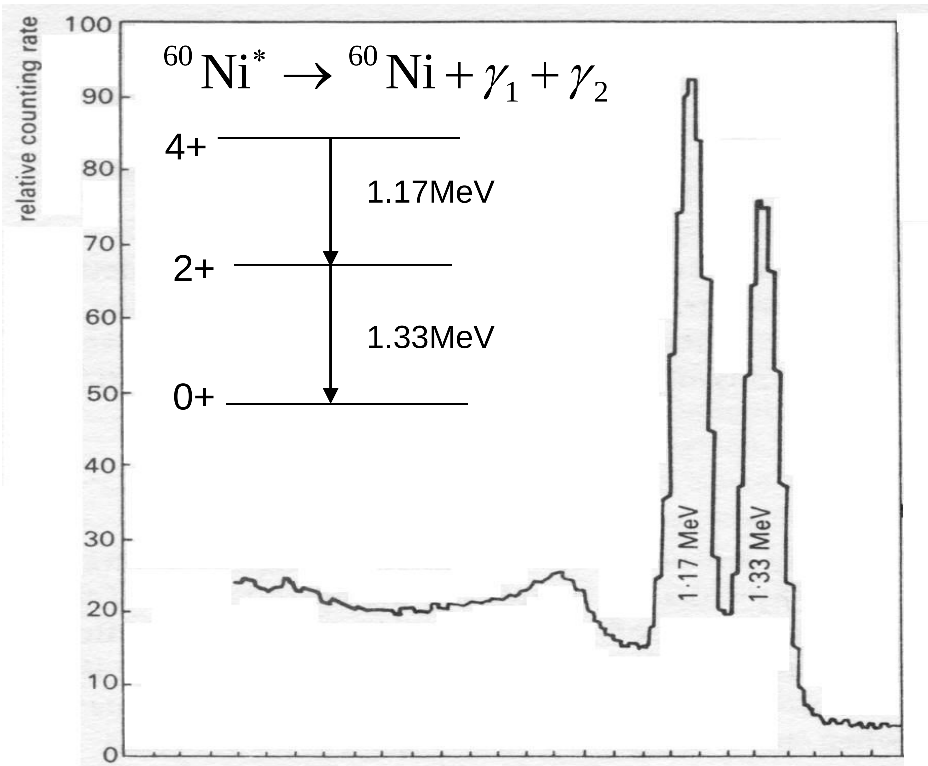
BETA PLUS DECAY



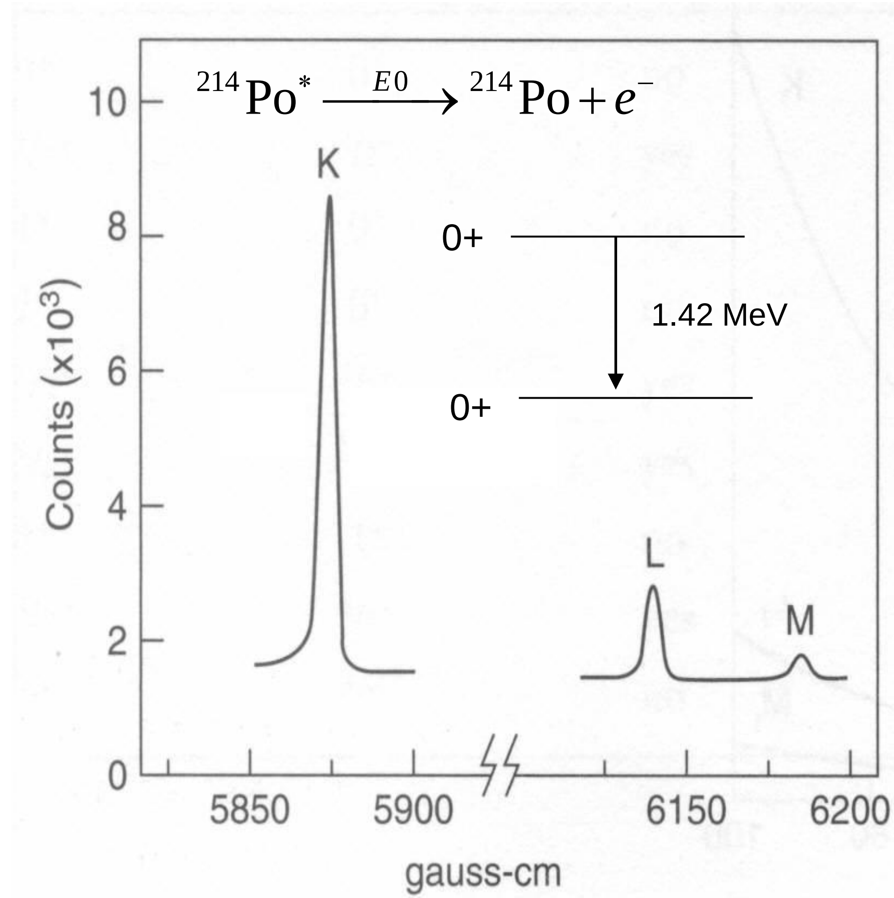
PAIR INTERNAL CONVERSION

Note how similar the spectral shapes are for positron emission even though the BETA PLUS is via the WEAK force, while PAIR INTERNAL is via the EM force. This is **because in the final state there are 3 PARTICLES** (**Daughter nucleus + 2 Leptons**).

Beta and Gamma similarities



GAMMA DECAY



INTERNAL CONVERSION

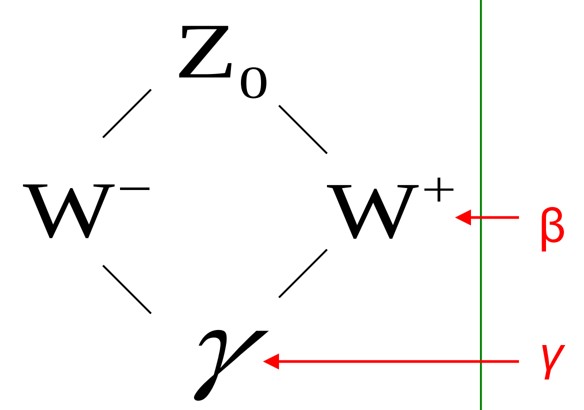
GAMMA decay and INTERNAL CONVERSION decay both show **discrete lines** – WHY because these are 2 body decays. What about data for ELECTRON CAPTURE – well that would require looking at the energy spectrum of emitted neutrinos – something not yet achieved (Why?).

Gamma and Beta decays are similar

Because the force carriers all have $J^{\pi} = 1^{-}$ the basic decays are similar in terms of angular momentum:.. NO MORE THAN **ONE UNIT OF ANG. MOMENTUM**.

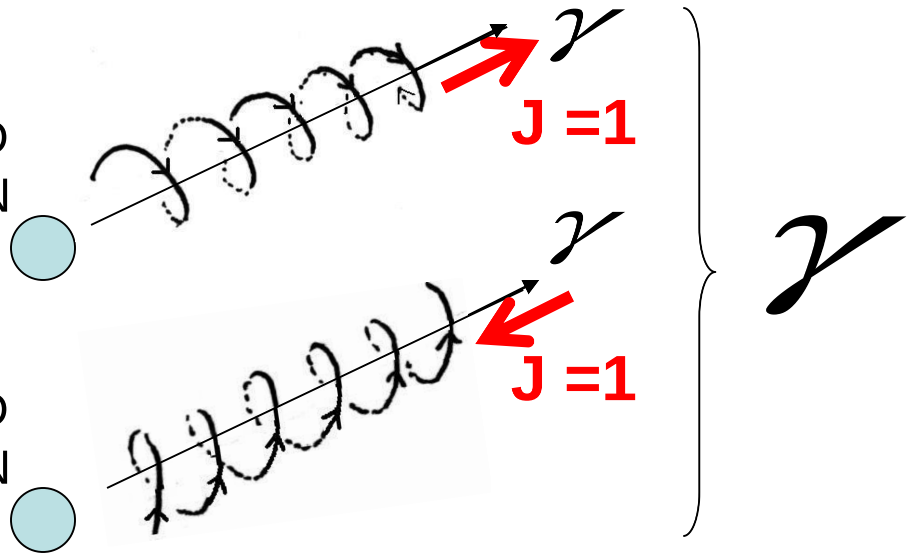
FAMILY CHARACTERISTIC

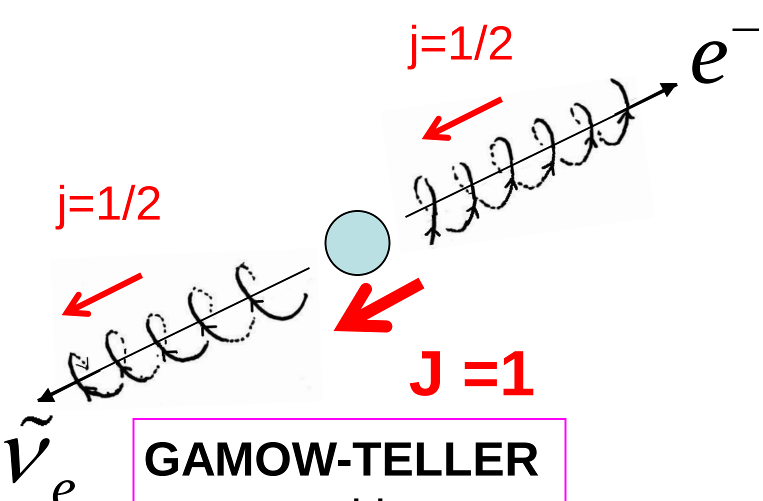
$J^{\pi} = 1^{-}$



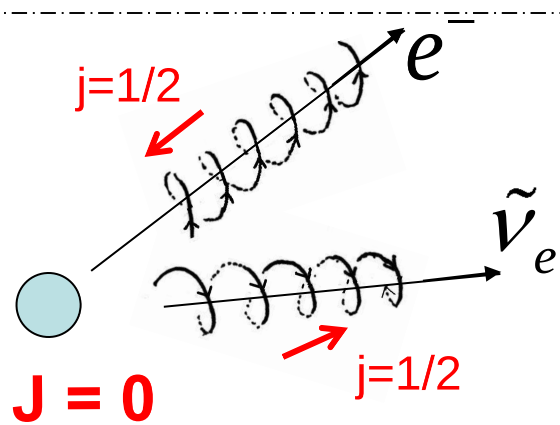
RIGHT HANDED PHOTON

LEFT HANDED PHOTON





GAMOW-TELLER transition



FERMI transition

LEFT HANDED ELECTRON

RIGHT HANDED ANTI - NEUTRINO

β^-

Gamma and Beta decays are similar

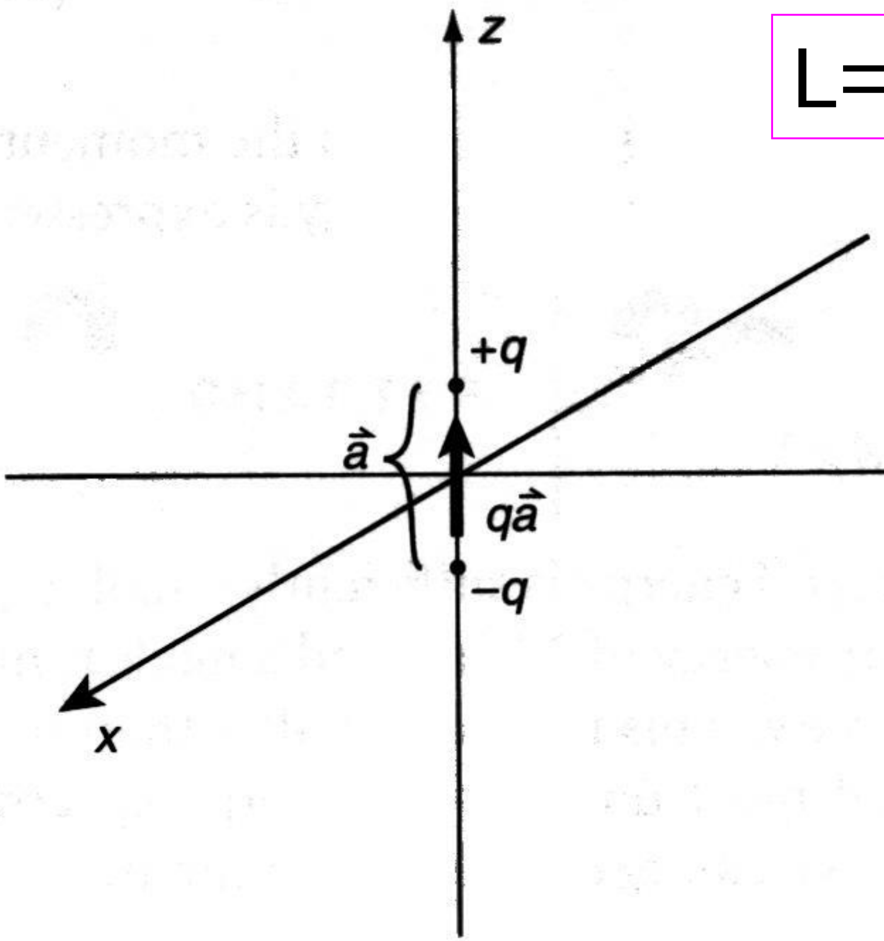
Decay		L	ΔJ	Nuclear Parity Change
electric dipole	E1	1	±1	yes
magnetic dipole	M1	1		no
electric quadrupole	E2	2	±2	no
magnetic quadrupole	M2	2		yes
electric octupole	E3	3	±3	yes
magnetic octupole	M3	3		no
electric hexadecapole	E4	4	±4	no
magnetic hexadecapol	M4	4		yes

γ

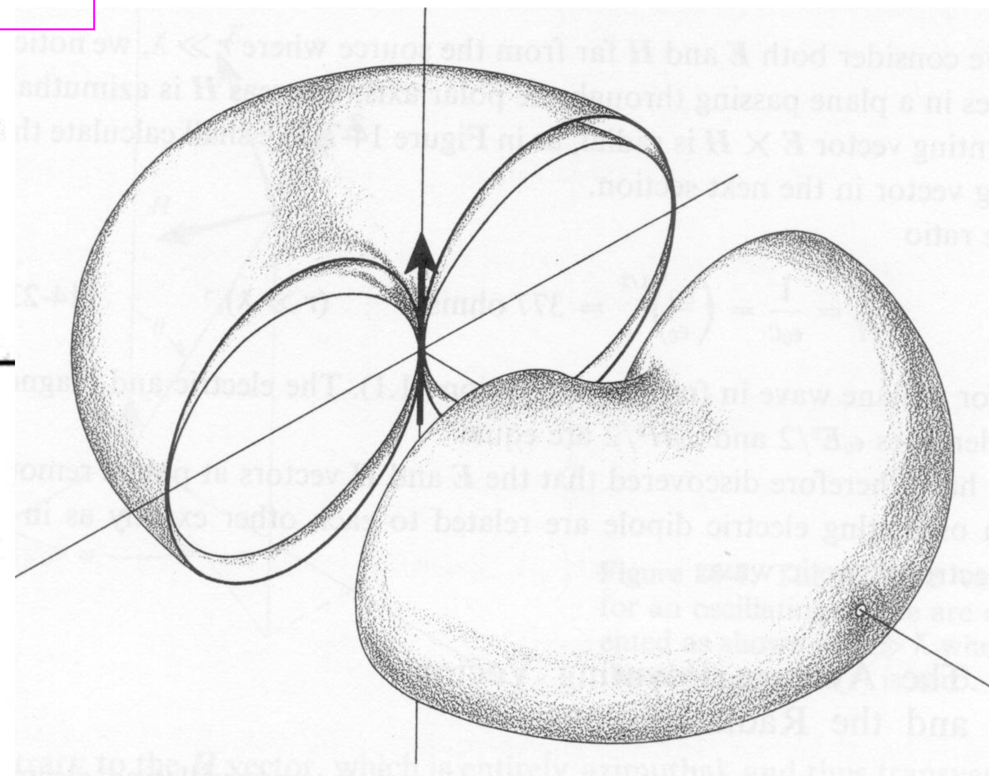
Decay	L	ΔJ	Nuclear Parity Change
allowed	0	0, ±1	no
1st forbidden	1	0, ±1, ±2	yes
2nd forbidden	2	±1, ±2, ±3	no
3rd forbidden	3	±2, ±3, ±4	yes
4th forbidden	4	±3, ±4, ±5	no

β

Electric Dipole (E1) Radiation



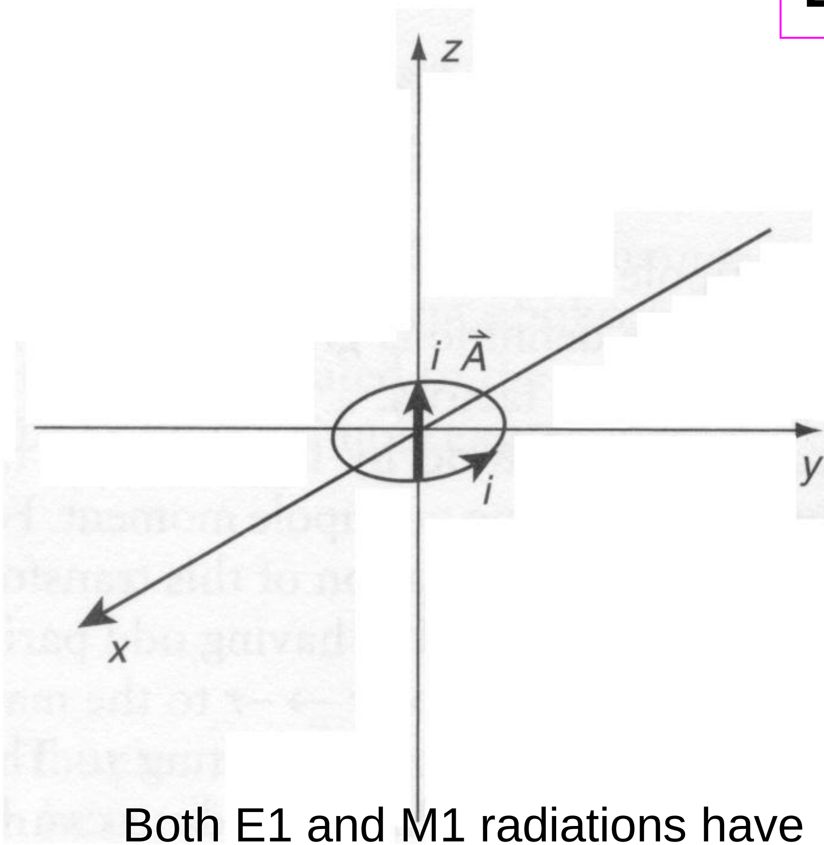
L=1

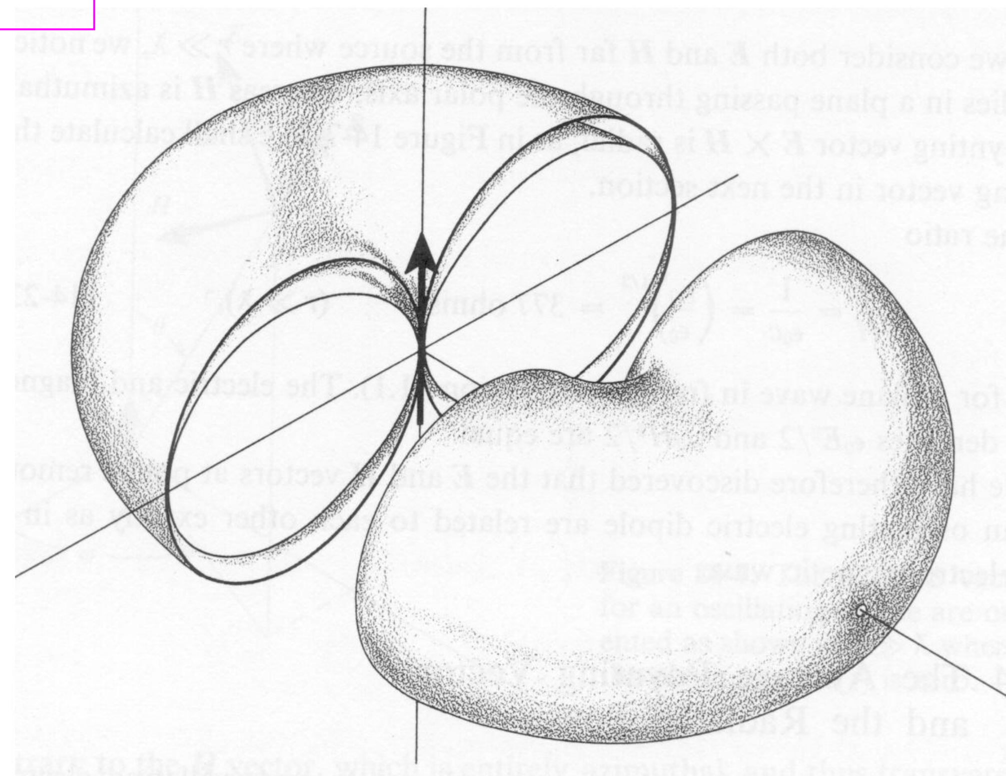


Magnetic Dipole (M1) Radiation

But an oscillating magnetic dipole gives exactly the same radiation pattern.

L=1



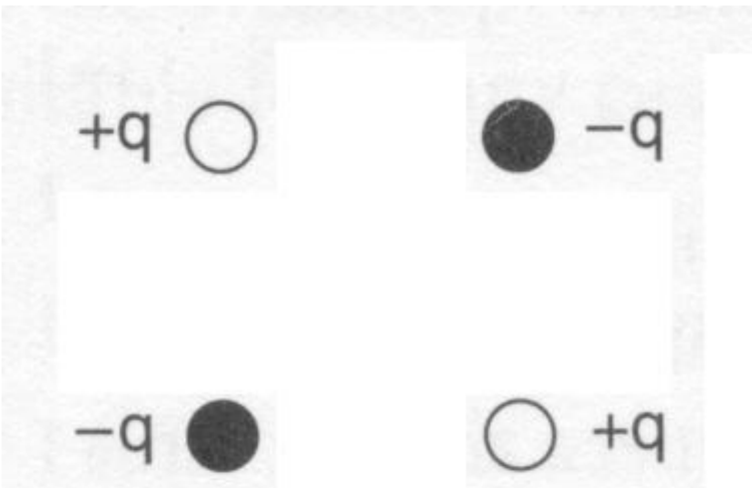


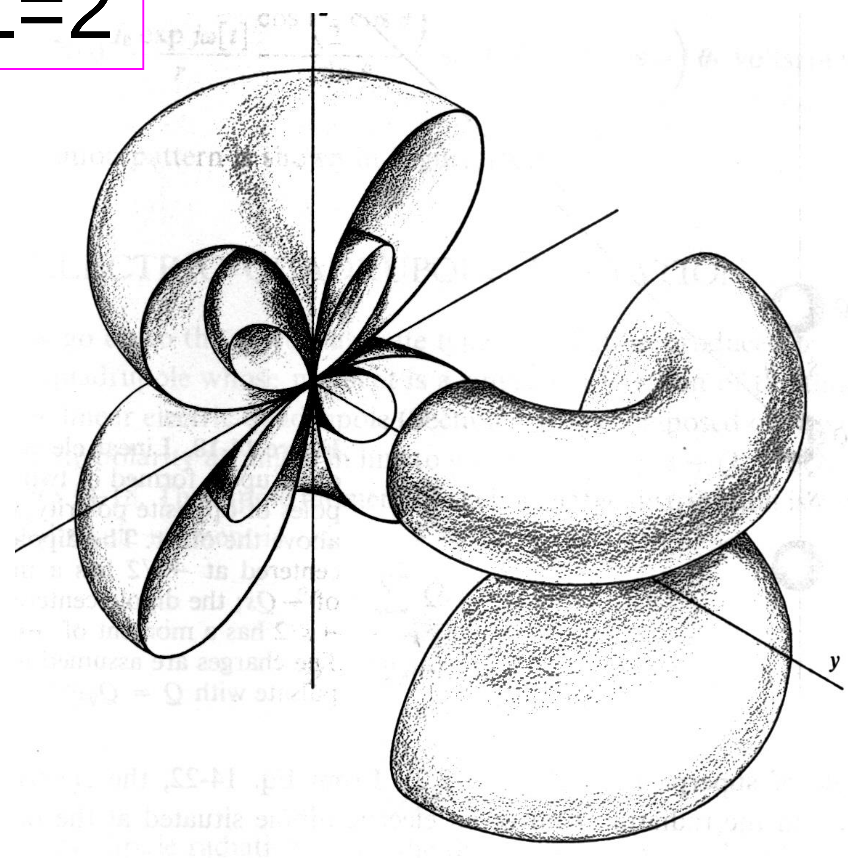
Both E1 and M1 radiations have **L=1** which means that this sort of radiation carries with it ONE unit of angular momentum. The distribution on the right is the probability of photons being emitted.

1st Forbidden Transitions are **L=1** and have this same emission pattern.

Electric Quadrupole (E2) Radiation

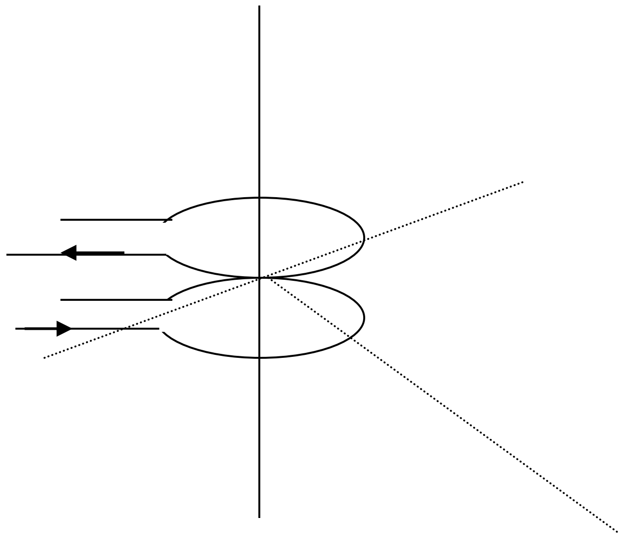
L=2

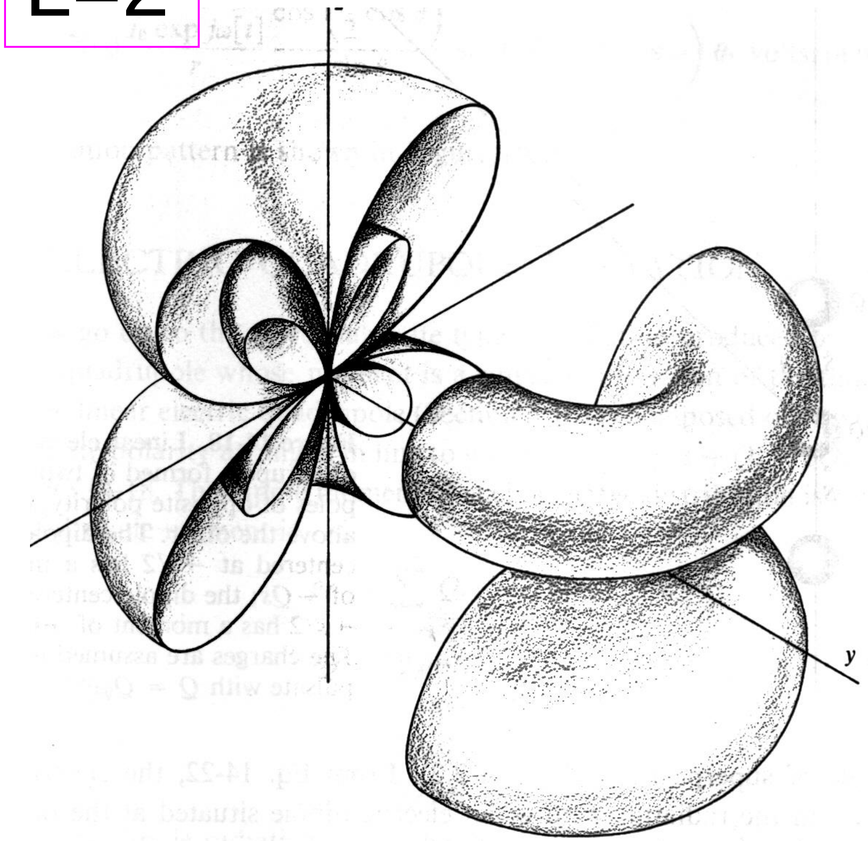




Magnetic Quadrupole (M2) Radiation

L=2

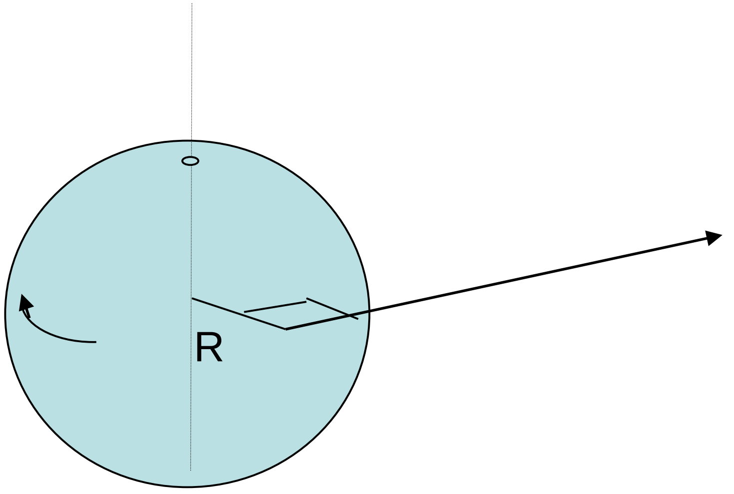




Both E2 and M2 are L=2 radiations – i.e. the photons carry away with them 2 units of angular momentum. The diagram on the right shows the directional probability distribution of photons. This same distribution applies to 2nd forbidden transitions. How is **L= 2** possible? **Doesn't the photon only have only one unit of angular momentum? In Beta decay isn't J=1 for the lepton pair the maximum?**

EMISSION FROM HIGHER MULTIPOLES IS DIFFICULT BUT NOT IMPOSSIBLE. QUANTUM MECHANICS ALLOWS IT.

Imagine that the nucleus wants to get rid of 1 unit of ang. mom. by emitting a photon from its surface. The maximum ang. mom that can be transferred is:



$$p = \frac{E_\gamma}{c}$$

$$\Delta J = p.R = \left(\frac{ER}{c}\right) = \left(\frac{ER}{\hbar c}\right).\hbar \qquad (8)$$

where p and E are the momentum and energy of the outgoing lepton pair wave. Putting E=1MeV (typical decay energy) and R=5F (typical nuclear radius) we get

$$\left(\frac{E_\gamma R}{\hbar c}\right) = \left(\frac{1MeV.5F}{197MeV.F}\right) \sim 0.025$$

i.e. classically one could only get 1/40 of an ang.mom unit. Quantum mechanics allows tunneling to larger distances – but the transition rates are reduced by factor:

$$(kR)^2 = \left(\frac{p}{\hbar}R\right)^2 = \left(\frac{ER}{\hbar c}\right)^2 \sim 5x10^{-4}$$

The FERMI GOLDEN RULE

Beta and Gamma decay are also alike in that the transition (decay) rate for their transitions can be calculated by the FERMI GOLDEN RULE. You can read about its derivation from the Time Dependent Schrodinger equation in most QM books

Enrico Fermi
(1901-1954)

It gives the rate of any decay process. It finds easy application to BETA and GAMMA decay. Fermi developed it and used it in 1934 in his THEORY OF BETA DECAY that is still the basic theory used today.

It looks as follows:

$$\lambda = \frac{2\pi}{\hbar} \left| < f \left| H \right| i > \right|^2 \frac{dn_f}{dE}$$

λ = the decay rate:

$\langle f | H | i \rangle$ = the "matrix element" or the overlap between initial state and final state via the Force interaction H

$\frac{dn_f}{dE}$ = the "density of states". The more available final states the faster the decay will go.

THEORY OF GAMMA DECAY

(i) MATRIX ELEMENT   This is difficult to obtain without a full knowledge of quantum field theory but we can quote the result.

Daughter wavefunction

Photon wavefunction

Parent wavefunction

$$< f\left|H_{EM}\right|i >= H_{if} = \int \psi_D * \psi_\gamma * \hat{H}_{EM} \psi_P d^3\vec{r}$$

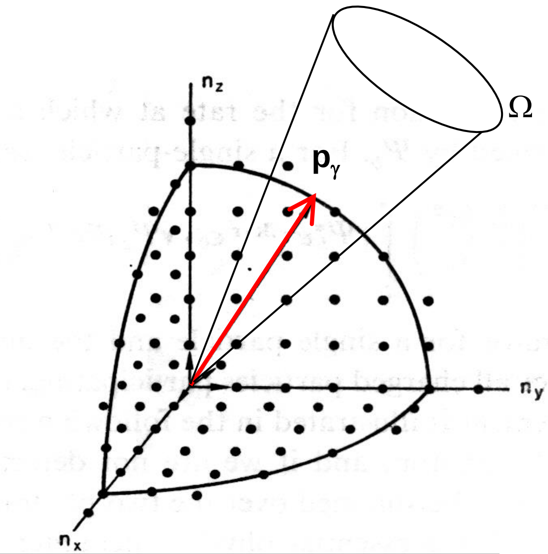
$$= \sqrt{\frac{2\pi}{V}} .\alpha.(\hbar c)(\hbar\omega).\int \psi_D * e^{-i\vec{k}.\vec{r}} (\hat{\boldsymbol{\varepsilon}}.\mathbf{r}) \psi_P d^3\vec{r}$$

(ii) DENSITY OF STATES. The density in phase space is a constant so that:

$$dN = \frac{V}{(2\pi)^3} d^3k_\gamma = \frac{V}{(2\pi)^3} .4\pi k_\gamma^2 dk_\gamma$$

Where V is the volume of a hypothetical box containing the nucleus. Because emission is directional we must work per unit solid angle:

$$\frac{dN}{d\Omega} = \frac{V}{(2\pi)^3} k_\gamma^2 dk_\gamma = \frac{V}{(2\pi)^3} \frac{p_\gamma^2 dp_\gamma}{\hbar^3}$$

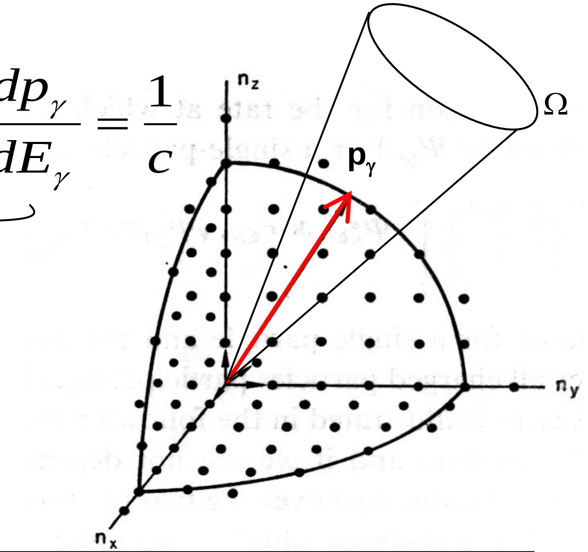


photon momentum space

THEORY OF GAMMA DECAY

Now lets write down the no of states per solid angle and per unit energy (remember $E_\gamma = p_\gamma c$, or $\frac{dp_\gamma}{dE_\gamma} = \frac{1}{c}$

$$\frac{d^2N}{d\Omega . dE_\gamma} = \frac{V}{(2\pi)^3} \frac{p_\gamma^2}{\hbar^3} \frac{dp_\gamma}{dE_\gamma} = \frac{V}{(2\pi)^3} \frac{E_\gamma^2}{(\hbar c)^3}$$



photon momentum space

(iii) APPLY THE FERMI GOLDEN RULE

$$d\lambda_\gamma = \frac{2\pi}{\hbar} \left| < f \left| H \right| i > \right|^2 \frac{dn_f}{dE}$$

$$= \frac{2\pi}{\hbar} . \frac{2\pi}{V} \alpha . (\hbar c)(\hbar \omega) M_{if}^2 . \frac{V}{(2\pi)^3} \frac{(E_\gamma)^2}{(\hbar c)^3} d\Omega$$

$$d\lambda_\gamma = \frac{1}{(2\pi)} . \alpha c . \left(\frac{E_\gamma}{\hbar c} \right)^3 M_{if}^2 d\Omega$$

Where:

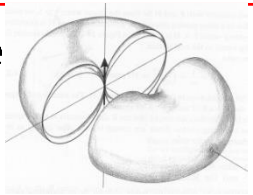
$$M_{if} = \int \psi_D^* e^{-i\mathbf{k}.\mathbf{r}} (\hat{\boldsymbol{\varepsilon}}.\mathbf{r}) \psi_P d^3 r$$

This is the PARTIAL DECAY RATE: I.E. the decay rate into an solid angle $d\Omega$ in a certain direction as determined by M_{if}

THEORY OF GAMMA DECAY

(iv) INTEGRATE **OVER ANGLE** TO GET THE TOTAL DECAY RATE

- because the decay rate varies with angle



$$\lambda_\gamma = \frac{1}{(2\pi)}.\alpha c.\left(\frac{E_\gamma}{\hbar c}\right)^3 \int_\theta M_{if}^2 .d\Omega$$

$d\Omega$ ← $2\pi.\sin\theta d\theta$

M_{if}^2 ← $\cos^2\theta$ dependence

$$\lambda_\gamma = \frac{1}{3}.\alpha c.\left(\frac{E_\gamma}{\hbar c}\right)^3 \mathrm{D_{if}}^2$$

$$\mathrm{D}_{if} = \int \psi_D^* z \psi_P d^3 r =$$ Nuclear dipole matrix element

---

More generally this result can be extended to higher angular momentum (L) radiation components:

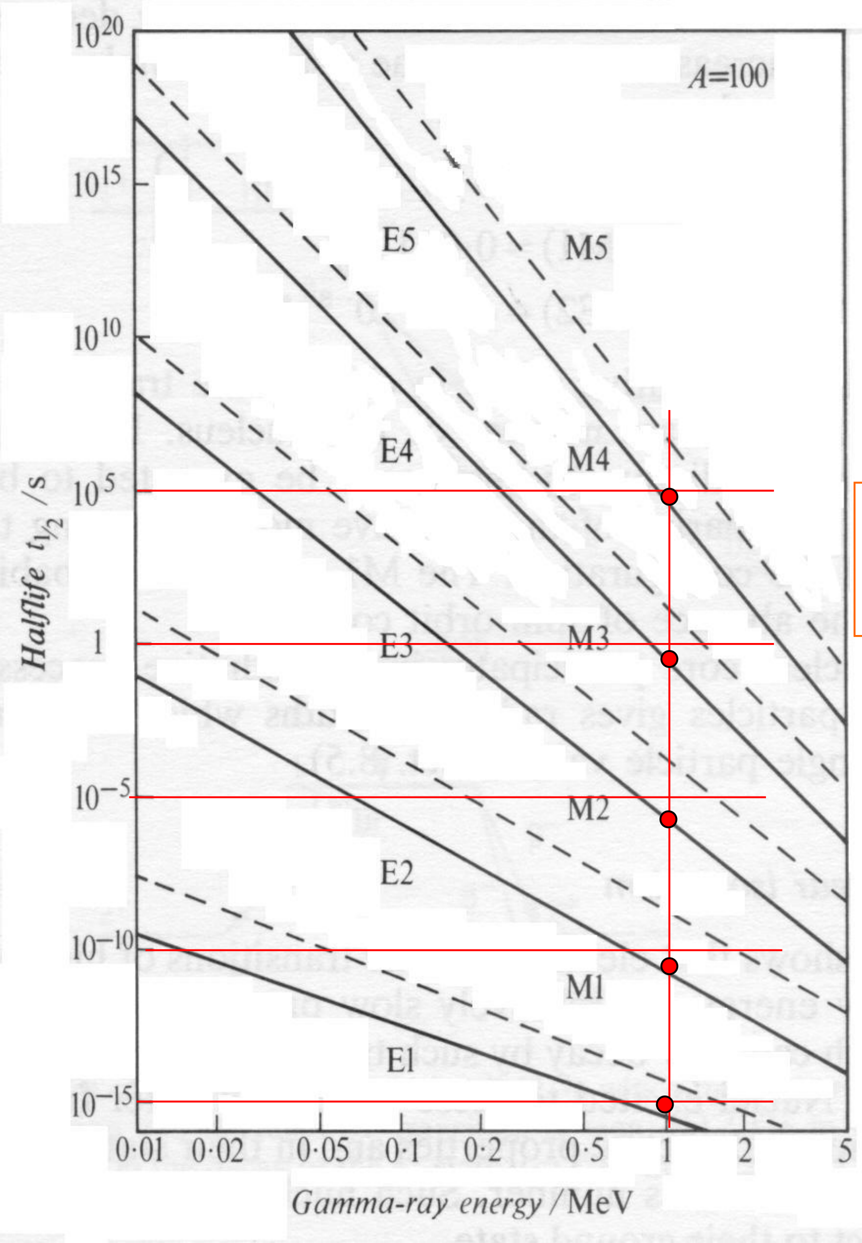
$$\lambda_\gamma(EL) = \frac{2(L+1)}{L\left[(2L+1)!!\right]^2}.\alpha c.\left(\frac{E_\gamma}{\hbar c}\right)^{2L+1} \left|Q_{if}(L)\right|^2$$ Eqn. 10.16

$$Q_{if}(L) = \int \psi_f^* Q(L) \psi_i d^3 r$$

is the matrix element of the multipole operator *Q*(L).

The Weisskopf Estimates

The lines show the half-life against gamma decay $\tau_{1/2} = \dfrac{0.692}{\lambda_\alpha}$



For what are known as the **Weisskopf Estimates** named after Victor Weisskopf who showed using the single particle shell model (1963) that (Eqns 10:20, 10 :22):

$$Q_{if} = \left[\frac{3}{L+3}\right] R^{2L}$$

Giving:

$$\lambda_\gamma(EL) = \frac{2(L+1)}{L[(2L+1)!!]^2}\left[\frac{3}{L+3}\right]^2 .\alpha c.\left(\frac{E_\gamma}{\hbar c}\right)^{2L+1} R_0^{2L} A^{2L/3}$$

$$\lambda_\gamma(ML) = 0.308A^{-2/3}.\lambda_\gamma(EL) \approx 10^{-2}.\lambda_\gamma(EL)$$

One can work out transition speeds approximately by scaling the result that

$$\tau_{1/2}(EL, A=100, E_\gamma = 1MeV) \approx 10^{5L-20} s$$

and noting that the rate scales as $A^{2L/3}$

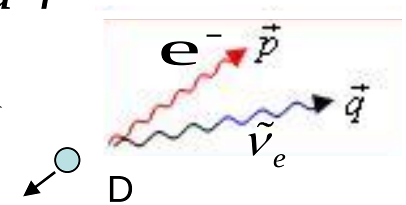
and as E_γ^{2L+1} - speeding up with both **size** and **energy.**

THEORY OF BETA DECAY

(i) MATRIX ELEMENT As with EM decay we write:

$$< f\,|H_{weak}|\,i >= H_{if} = \int \psi_D{}^* \psi_e{}^* \psi_{\nu_e}{}^* \hat{H}_{Weak} \psi_P d^3\vec{r}$$

$$= G\int \psi_D{}^* \psi_e{}^* \psi_{\nu_e}{}^* .\psi_P d^3\vec{r}$$



This follows because the 4 point vertex is close to being a "point" interaction. G is a constant representing the strength of the weak interaction. The outgoing electron and neutrino waves into imaginary volume V are written

$$\psi_e(r) = \frac{1}{\sqrt{V}} e^{i\vec{p}.\vec{r}/\hbar} \quad \text{and} \quad \psi_\nu(r) = \frac{1}{\sqrt{V}} e^{i\vec{q}.\vec{r}/\hbar}$$

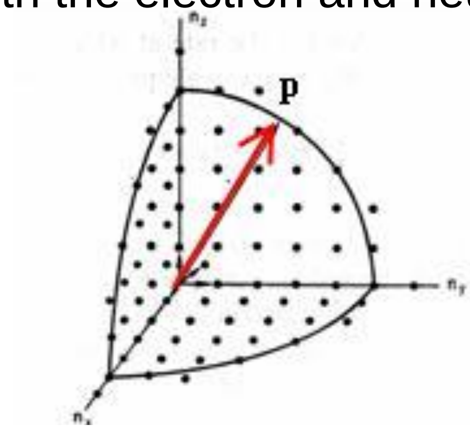
$$H_{if} = \frac{G}{V}\int \psi_D{}^* .e^{\frac{-i(\vec{p}+\vec{q}).\vec{r}}{\hbar}} \psi_P .d^3\vec{r} = \frac{GM_{if}}{V}$$

where $$M_{if} = \int \psi_D{}^* .e^{\frac{-i(\vec{p}+\vec{q}).\vec{r}}{\hbar}} \psi_P .d^3\vec{r} \approx \int \psi_D{}^* \psi_P .d^3\vec{r}$$

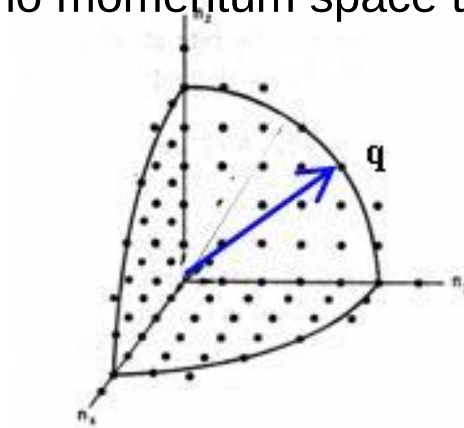
and: $$G = 8.8x10^{-5} MeV.F^3 = \beta - \text{decay coupling constant}$$

THEORY OF BETA DECAY

(ii) DENSITY OF STATES. The density is more complex in beta decay because there are both the electron and neutrino momentum space to consider:



Electron mom. space



Anti-neutrino mom. .space

The number of final states is a product of both phase space probabilities.

$$dn_f = dN_e . dN_\nu = \frac{V}{(2\pi)^3} d^3k_e . \frac{V}{(2\pi)^3} d^3k_\nu = \frac{V^2}{2\pi^4\hbar^6} p^2 q^2 dp dq$$

$$\therefore \frac{dn_f}{dE_f} = \frac{V^2}{2\pi^4\hbar^6} p^2 q^2 \frac{dq}{dE_f} dp$$

$$\frac{dn_f}{dE_f} = \frac{V^2}{4\pi^4\hbar^6 c^3} . (Q - T_e)^2 p^2 dp$$

Note that: $T_e + qc = Q_\beta$

i.e. that $q = \frac{Q_\beta - T_e}{c}$

So that with T_e fixed $\frac{dq}{dE_f} = \frac{1}{c}$

THEORY OF BETA DECAY

(iii) APPLY THE FERMI GOLDEN RULE

$$d\lambda = \frac{2\pi}{\hbar}\left|< f\left|H\right|i >\right|^2 \frac{dn_f}{dE}$$

$$= \frac{2\pi}{\hbar} . \frac{G^2 {M_{if}}^2}{V^2} . \frac{V^2}{4\pi^4\hbar^6 c^3} .(Q - T_e)^2 p^2 dp$$

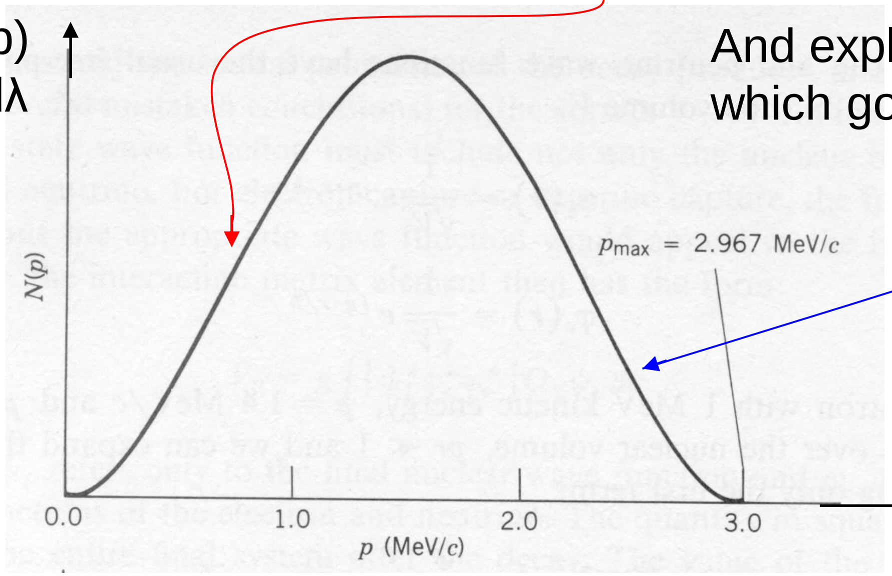
$$= \frac{G^2 {M_{if}}^2}{2\pi^3\hbar^7 c^3} (Q - T_e)^2 p^2 dp$$

Which explains the rise here which goes as p^2

N(p)
~ dλ

And explains the drop here which goes as $(Q-T_e)^2$

THE BETA SPECTRUM SHAPE

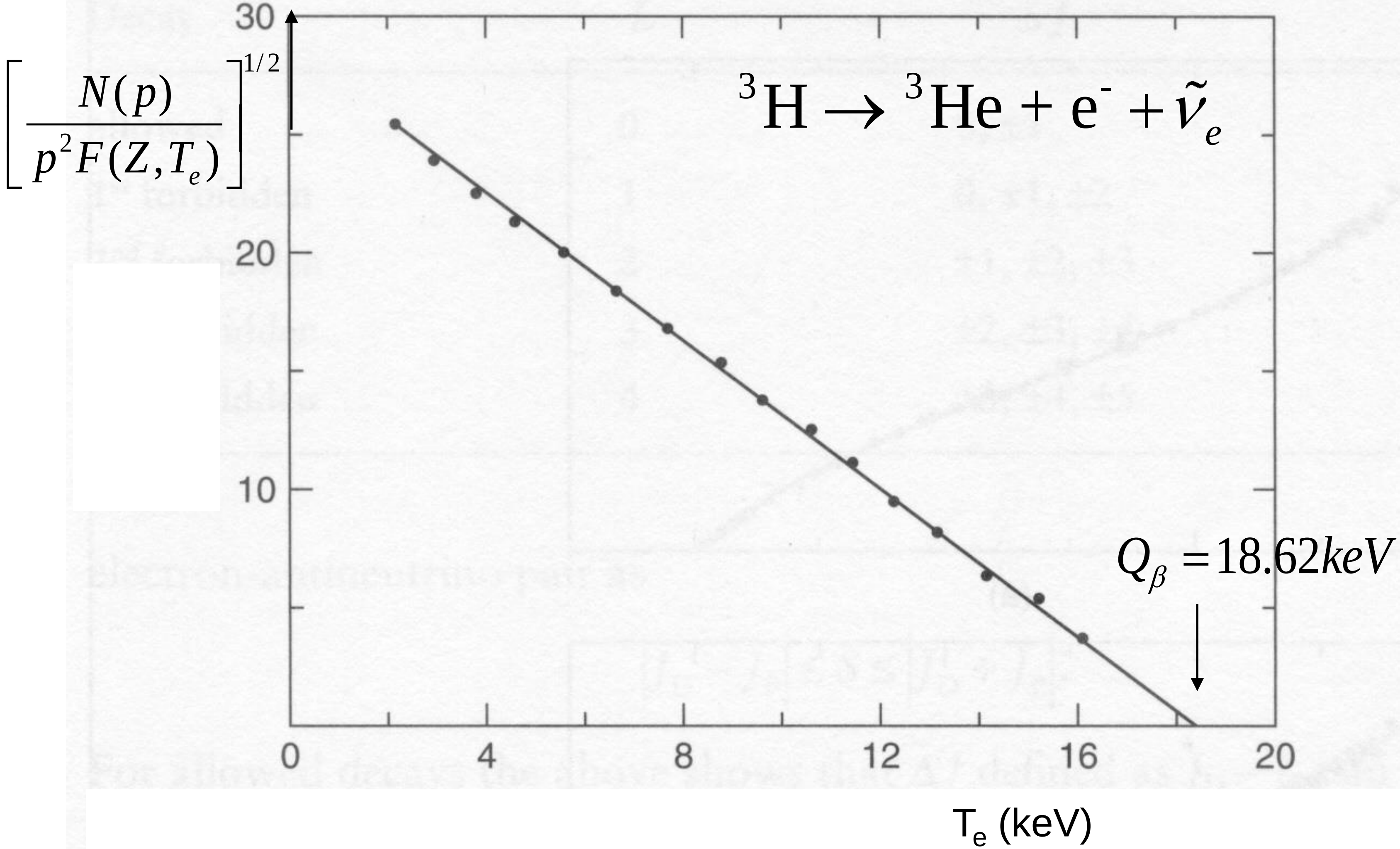


Electron mom. *p*

The Fermi-Kurie Plot

The Fermi-Kurie plot is very famous in β decay. It relies upon the shape of the β spectrum – i.e. that $N(p) = \text{Const.}p^2(Q_\beta - T_e)^2 F(Z,T_e)$

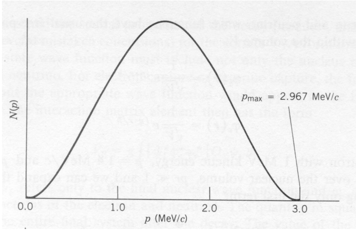
$F(Z,T_e)$ is the "Fermi factor " that deals with the distortion of the wavefunctions due to repulsion and attraction of the e^+ or e^- on leaving the nucleus.



THEORY OF BETA DECAY

(iv) INTEGRATE **OVER ENERGY** TO GET THE TOTAL DECAY RATE

Unlike gamma decay where we needed to integrate over angle – in BETA DECAY we have to INTEGRATE OVER ENERGY in order to get the total decay rate. [For L=0 the there is no directional emission pattern]



$$\lambda_\beta = \int_0^{p_{\max}} N(p)dp = c.\frac{G^2(m_e c^2)^5}{2\pi^3(\hbar c)^7}.\mathrm{f}(Z,Q_\beta)$$

where **f**(Z,Q) is the Fermi integral

$$\mathrm{f}(Z_D,Q_\beta) = \frac{1}{(m_e c^2)^5}\int_0^{p_{\max}} F(Z_D,T_e)(pc)^2(Q_\beta - T_e)^2 d(pc)$$

$$T_e = \sqrt{(pc)^2 + (m_e c^2)^2} - m_c c^2$$

which is a very complex integral – needing calculation by computer. The values can be found in nuclear tables. But there is one solution if $Q_\beta \gg m_e c^2$ then -

$$\mathrm{f}(Z_D,Q_\beta) = \frac{1}{30}\left[\frac{Q_\beta}{m_e c^2}\right]^5$$

$$\lambda \approx c.\frac{G^2\left|M_{if}\right|^2}{60\pi^3(\hbar c)^7}.Q_\beta^5$$

Sargent's Law