Surface, Volume Integrations and Related Theorems

Surface Integrations

Let $\vec{F}$ be a vector defined in a region bounded by a surface S. The surface integral of $\vec{F}$ over the surface S is defined by

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_S \vec{F}\cdot d\vec{s}$$

where, $\hat{n}$ is the unit normal vector (outward drawn) on the elementary surface ds and $d\vec{s} = \hat{n}\,ds$

e.g. – Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and S is the part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Ans:

$\hat{n}$ = unit normal vector to S

$$= \frac{\text{grad}(x^2+y^2+z^2)}{|\text{grad}(x^2+y^2+z^2)|} = \frac{2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \iint \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \qquad \Big|\ |\hat{n}\cdot\hat{k}| = z$$

$\vec{F}\cdot\hat{n} = 3xyz$

~~So~~ The first octant is bounded by x axis $(y=0)$ and y axis $(x=0)$ and $x^2+y^2=1$, $z=0$

$$\int_S \vec{F}\cdot\hat{n}\,ds = 3\int_0^1 x\left\{\int_0^{\sqrt{1-x^2}} y\,dy\right\}dx = \frac{3}{8} \text{ sq. units}$$

Scanned by CamScanner

Gauss' Divergence Theorem

Let $\vec{F}$ be a continuously differentiable vector point function ~~and~~ defined in a region V and S be a closed surface enclosing the volume of V. Then

$$\int_V (\vec{\nabla}\cdot\vec{F})\, dV = \oint_S \vec{F}\cdot\hat{n}\, ds$$

where $\hat{n}$ is the unit outward drawn normal vector

Stoke's Theorem

Let $\vec{F}$ be a continuously differentiable vector point function defined on a surface S bounded by a curve Γ. Then

$$\int_\Gamma \vec{F}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds$$

where $\hat{n}$ is the unit outward drawn normal vector

In Cartesian form [Let $\vec{F} = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$]

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S \left(F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy\right)$$

[Gauss Th]

$$\int_\Gamma \left(F_1\, dx + F_2\, dy + F_3\, dz\right) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\, dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz\, dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dy\, dx\right]$$

[Stoke's Th]

Scanned by CamScanner

Green's Theorem

Let R be a closed region of the xy-plane bounded by a simple closed curve Γ and if P and Q are continuous functions of x and y having continuous derivatives in R, then Green's theorem states

$$\oint_{\Gamma} P\,dx + Q\,dy = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

In vector notation

Let $\vec{F} = P\hat{i} + Q\hat{j}$ and Γ be a closed curve surrounding a region R of the xy-plane. If $\hat{n}$ be the outward drawn unit normal to Γ, then ~~show that the~~ show that

$$\oint_{\Gamma} \vec{G}\cdot\hat{n}\,ds = \iint_{R} (\vec{\nabla}\cdot\vec{G})\,dR$$

where, $\vec{G} = \vec{F} \times \hat{k}$

Proof - Left as exercise

Scanned by CamScanner