SOLUTION OF SAT-1

SEMESTER-IV (HONS.)- 2020

Subject: Mathematics

Course Code: MTMACOR10T

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1. Define sub-space of a vector space. Examine if the set $S = \{(x, y, z) \in R^3 : x + 2y - z = 1, 2x - y + z = 2$ is a subspace of $R3$ or not.

<u>SOLUTION</u>: For first Part see Higher Algebra Book by S.K.Mapa.

2nd Part: S is not subspace of R^3 since S does not contain the null vector.

2. Define linear span of a set. Prove that $L(S)$ is the smallest subspace of vector space V containing the set S.

<u>SOLUTION</u>: See Higher Algebra Book by S.K.Mapa Page 123 and 124, Theorem 4.3.9.

3. Show that a linearly independent set of vectors of a finite dimensional vector space V over a field F is either a basis of V or can be extended to a basis of V. Extend the set $(1,1,-1), (2,1,1)$ to a basis of R^3.

<u>SOLUTION</u>: See Higher Algebra Book by S.K.Mapa Page 140 Theorem 4.5.7. (Extension theorem).

2nd Part:

Let $\alpha_1 = (1,1,-1)$, $\alpha_2 = (2,1,1)$.

Let us examine if $\alpha_3 = (0,0,1)$ belongs to $L\{\alpha_1, \alpha_2\}$.

Let $\alpha_3 = c_1\alpha_1 + c_2\alpha_2$, where $c_1, c_2 \in R$

Then $(0,0,1) = c_1(1,1,-1) + c_2(2,1,1)$

$= (c_1 + 2c_2, c_1 + c_2, -c_1 + c_2)$

$\therefore$ $c_1 + 2c_2 = 0$, $c_1 + c_2 = 0$, $-c_1 + c_2 = 1$

From 1st and 2nd equ^n, on subtraction we get $c_2 = 0$.

Again from 2nd and 3rd equation on adding we get $c_2 = 1/2$

$\therefore$ This is an inconsistent system of equations in c_1, c_2. Therefore $\alpha_3 \notin L\{\alpha_1, \alpha_2\}$

Consequently, the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent set in R^3.

Since R^3 is a vector space of dimension 3 and the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent set of 3 vectors in R^3, $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of R^3. <u>Ans</u>

4. Prove that a set of vectors containing the null vector in a vector space V(F) is linearly dependent.

<u>SOLUTION</u>: See Higher Algebra Book by S.K.Mapa Page 127 Theorem 4.4.3.

5. Let $W = \{(x, y, z) \in R^3 : x - 4y + 3z = 0\}$. Show that W is a subspace of R^3. Find the dimension of W.

<u>SOLUTION</u>:

W is a non-empty subset of R^3, since $(0,0,0) \in W$.

Let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2) \in S$; $x_i, y_i, z_i \in R$, $i = 1, 2$.

~~let $c, d \in R$.~~ Then $x_1 - 4y_1 + 3z_1 = 0$ and $x_2 - 4y_2 + 3z_2 = 0$ $\}$ — (1)

Let $c, d \in R$. Then $c\alpha + d\beta = c(x_1, y_1, z_1) + d(x_2, y_2, z_2)$
$= (cx_1 + dx_2,\ cy_1 + dy_2,\ cz_1 + dz_2)$

Now $(cx_1 + dx_2) - 4(cy_1 + dy_2) + 3(cz_1 + dz_2)$
$= c(x_1 - 4y_1 + 3z_1) + d(x_2 - 4y_2 + 3z_2)$
$= c \cdot 0 + d \cdot 0$ (using the system of equ^n (1))
$= 0$

$\therefore$ $c\alpha + d\beta \in W$. This proves that W is a subspace of R^3.

Let $\xi = (a, b, c) \in W$. Then $a, b, c \in R$ and $a - 4b + 3c = 0$

Therefore $\xi = (4b - 3c,\ b,\ c) = b(4,1,0) + c(-3,0,1)$.

Let $\alpha = (4,1,0)$, $\beta = (-3,0,1)$. Then $\xi = b\alpha + c\beta \in L\{\alpha, \beta\}$.

Therefore $W \subset L\{\alpha, \beta\}$.

Again $\alpha \in W$, $\beta \in W$. This implies $L\{\alpha, \beta\} \subset W$, as W is a subspace.

Consequently, $W = L\{\alpha, \beta\}$.

α, β are linearly independent in W, since $c_1\alpha + c_2\beta = 0 \Rightarrow 4c_1 - 3c_2 = 0,\ c_1 = 0,\ c_2 = 0$.

$\Rightarrow c_1 = 0 = c_2$. Hence the set $\{\alpha, \beta\}$ is a basis of W and $\dim W = 2$. Ans.

6. If V be the real vector space of all real matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and W be the subset of those matrices of V for which $a + b = 0$, then prove that W is a subspace of V and find a basis of W.

SOLUTION:

$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a, b, c, d \in R \right\}$ and $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \ \&\ a + b = 0 \right\}$

We note that $W \neq \phi$ since $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

Let us assume that $\alpha = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $\beta = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in W$ and $\lambda \in R$.

then $a_1 + b_1 = 0$ and $a_2 + b_2 = 0$.

Now $\alpha + \beta = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} \in W$ since $a_1 + a_2 + b_1 + b_2 = a_1 + b_1 + a_2 + b_2 = 0$

and $\lambda\alpha = \lambda \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} \lambda a_1 & \lambda b_1 \\ \lambda c_1 & \lambda d_1 \end{pmatrix} \in W$ since $\lambda a_1 + \lambda b_1 = \lambda(a_1 + b_1) = 0$

Thus W is a subspace of V.

Let $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$ Then $a + b = 0 \Rightarrow b = -a$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -a \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $S = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \in W$

Then $\xi \in L(S)$.

Let $\lambda_1 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \lambda_1 & -\lambda_1 \\ \lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Thus the set S is linearly independent.

Hence S is a basis of the subspace W. Ans.