● **Matrix Representation of Lin. map** :- $\dim V = n$, $\dim W = m$, $T: V \to W$ is linear. Let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis of V and $\{\beta_1, \ldots, \beta_m\}$ be an ordered basis of W. Then T is completly known, if we know $T(\alpha_1), \ldots, T(\alpha_n)$.

Let $T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \cdots + a_{m1}\beta_m$,

$T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \cdots + a_{m2}\beta_m$

$\cdots\cdots\cdots\cdots$

$T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \cdots + a_{mn}\beta_m$.

Then for $i = 1, \ldots, m$; $j = 1, \ldots, n$; a_{ij} is uniquely determined element of F.

Let $\xi = x_1\alpha_1 + \cdots + x_n\alpha_n \in V$ be arbitrary.

Let $T(\xi) = y_1\beta_1 + \cdots + y_m\beta_m$.

Now, $T(\xi) = x_1 T(\alpha_1) + \cdots + x_n T(\alpha_n)$.

$$\Rightarrow y_1\beta_1 + \cdots + y_m\beta_m = x_1(a_{11}\beta_1 + a_{21}\beta_2 + \cdots + a_{m1}\beta_m)$$
$$+ x_2(a_{12}\beta_1 + a_{22}\beta_2 + \cdots + a_{m2}\beta_m)$$
$$+ \cdots + x_n(a_{1n}\beta_1 + \cdots + a_{mn}\beta_m).$$
$$= (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)\beta_1$$
$$+ (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n)\beta_2$$
$$+ \cdots + (a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n)\beta_m.$$

$$\Rightarrow y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\cdots\cdots\cdots$$
$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n.$$

i.e, $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = (a_{ij})_{m \times n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, i.e, $Y = AX$, where ①

$A = (a_{ij})_{m \times n}$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is the coordinate vector of an arbitrary element ξ of V w.r.t. the ordered basis $\{\alpha_1, \ldots, \alpha_n\}$ and $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ is the coordinate vector of $T(\xi)$ in W w.r.t. the ordered basis $\{\beta_1, \ldots, \beta_m\}$.

(1) is called the matrix representation of T relative to the chosen ordered bases of V and W.

If $\xi = \alpha_j$, then $x_1 = 0 = x_2 = \cdots = x_{j-1} = x_{j+1} = \cdots = x_n$, $x_j = 1$ and $y_1 = a_{1j}, y_2 = a_{2j}, \ldots, y_m = a_{mj}$. Thus the coordinate vector of $T(\alpha_j)$ w.r.t. the ordered basis $\{\beta_1, \ldots, \beta_m\}$ is the jth column of $A = (a_{ij})_{m\times n}$.

The matrix A is called the matrix associated with T or simply the matrix of T relative to the chosen ordered bases.

• **Prob!**– A lin. map $T: \mathbb{R}^3 \to \mathbb{R}^2$ is given by $T(x,y,z) = (3x-2y+z, x-3y-2z)$ $\forall (x,y,z) \in \mathbb{R}^3$. Find the matrix of T relative to the ordered bases

(i) $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$.

(ii) $\{(0,1,0), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1),(1,0)\}$ of $\mathbb{R}^2$.

Soln!– (i) $T(0,1,1) = (0-2+1, 0-3-2) = (-1,-5) = (-1)(1,0) + (-5)(0,1)$,

$T(1,0,1) = (3-0+1, 1-0-2) = (4,-1) = 4(1,0) + (-1)(0,1)$,

$T(1,1,0) = (3-2+0, 1-3-0) = (1,-2) = 1.(1,0) + (-2)(0,1)$.

$\therefore$ The matrix of $T = \begin{pmatrix} -1 & 4 & 1 \\ -5 & -1 & -2 \end{pmatrix}$.

• **Prob!**– The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0),(1,1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find T.

Soln!– $T(0,1,1) = 1(1,0) + 2(1,1) = (3,2)$,

$T(1,0,1) = 2(1,0) + 1(1,1) = (3,1)$

$T(1,1,0) = 4(1,0) + 0(1,1) = (4,0)$

Let $(x,y,z) \in \mathbb{R}^3$ be arbitrary. Let $(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$.

$c_1 - c_3 = z - x$
$c_1 + c_3 = y$
$z - \frac{y+z-x}{2}$

$3z + 2x + 2y - 2z$

Then $c_2 + c_3 = x$, $c_1 + c_3 = y$, $c_1 + c_2 = z$.

$\Rightarrow c_1 = \frac{y+z-x}{2}$, $c_3 = \frac{x+y-z}{2}$, $c_2 = \frac{z+x-y}{2}$

Now, $T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$

$= c_1(3,2) + c_2(3,1) + c_3(4,0) = (3c_1 + 3c_2 + 4c_3, 2c_1 + c_2)$

$= \left(2x+2y+z, \frac{y+3z-x}{2}\right)$, $\forall (x,y,z) \in \mathbb{R}^3$.

• **Th:**– Let V and W be finite dimensional vector spaces over a field F and $T: V \to W$ be a linear map. Then rank of T = rank of $m(T)$, where $m(T)$ is the matrix of T relative to any chosen ordered bases of V and W.

• **Th:**– Let $T: X \to Y$, $S: Y \to Z$ be linear maps, where X, Y, Z are finite dimensional vector spaces over a field F. Then relative to a choice of ordered bases of X, Y, Z, $m(ST) = m(S).m(T)$, where $m(T)$ is the matrix of T relative to the ~~chosen~~ chosen ordered bases.

• Pf:– Follow book (Mapa)

• **Th:-** Let V and W be finite dimensional vector spaces over a field F and $T: V \to W$ be a linear map. Then T is invertible iff the matrix $m(T)$ of T relative to any chosen ordered bases of V and W is non-singular.

Pf:- Follow book (Mapa)

• **Th:-** Let V and W be finite dimensional vector spaces of same dimension over a field F and $T: V \to W$ be an invertible linear map. If $m(T)$ be the matrix of T relative to a chosen ordered bases of V and W, then the matrix $m(T^{-1})$ of $T^{-1}: W \to V$ relative to the same ordered bases is given by
$$m(T^{-1}) = (m(T))^{-1}.$$

Pf:- Follow book (Mapa)

• **Probl:-** Let $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ be ordered bases of the real vector spaces V and W respectively. A linear map $T: V \to W$ maps the basis vectors as $T(\alpha_1) = \beta_1$, $T(\alpha_2) = \beta_1 + \beta_2$, $T(\alpha_3) = \beta_1 + \beta_2 + \beta_3$. Find the matrix $m(T)$ of T relative to these ordered bases. Show that T is invertible. Also find $m(T^{-1})$ relative to these ordered bases.

Soln:- $T(\alpha_1) = 1.\beta_1 + 0.\beta_2 + 0.\beta_3$, $T(\alpha_2) = 1.\beta_1 + 1.\beta_2 + 0.\beta_3$, $T(\alpha_3) = 1.\beta_1 + 1.\beta_2 + 1.\beta_3$.

$\therefore m(T) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Now $|m(T)| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$ so that $m(T)$ is non-singular. Therefore T is invertible.

Now, $T^{-1}(\beta_1) = \alpha_1$, $T^{-1}(\beta_1 + \beta_2) = \alpha_2$, $T^{-1}(\beta_1 + \beta_2 + \beta_3) = \alpha_3$.

i.e, $T^{-1}(\beta_1) = \alpha_1$, $T^{-1}(\beta_1) + T^{-1}(\beta_2) = \alpha_2$, $T^{-1}(\beta_1) + T^{-1}(\beta_2) + T^{-1}(\beta_3) = \alpha_3$.

$\Rightarrow T^{-1}(\beta_1) = 1.\alpha_1 + 0.\alpha_2 + 0.\alpha_3$,

$T^{-1}(\beta_2) = \alpha_2 - \alpha_1 = (-1)\alpha_1 + 1\alpha_2 + 0\alpha_3$

$T^{-1}(\beta_3) = \alpha_3 - \alpha_1 - (\alpha_2 - \alpha_1) = 0\alpha_1 + (-1)\alpha_2 + 1.\alpha_3$.

$\therefore m(T^{-1}) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Linear space of linear mappings:-

Let V and W be vector spaces over a field F. Let $T: V \to W$, $S: V \to W$ be linear maps. Define the sum $T+S$ and scalar multiplication cT $(c \in F)$ as: $T+S: V \to W$ is given by $(T+S)(\alpha) = T(\alpha)+S(\alpha), \forall \alpha \in V$.
$cT: V \to W$ is given by $(cT)(\alpha) = c(T(\alpha)), \forall \alpha \in V$.
Then it can be proved that $T+S$ and cT are linear maps.
Let L be the set of all linear maps from V to W. Then it can be proved that L is a linear space or vector space over the field F under the addition and scalar multiplication as defined above. The linear space of all linear maps from a vector space V to a vector space W over the same field F is denoted by $L(V, W)$. In particular, the linear space $L(V, F)$ is called the dual space of V.

- **Th:-** Let V and W be finite dimensional vector spaces over a field F with $\dim V = n$, $\dim W = m$. Let $M_{m,n}$ be the set of all $m \times n$ matrices over the field F. Then we know that $M_{m,n}$ is a vector space over the field F under matrix addition and matrix scalar multiplication. Then $L(V, W)$ is isomorphic to $M_{m,n}$.

Pf:- Follow book. (Mapa)

- **Th:-** Let V and W be finite dimensional vector spaces over the same field F with $\dim V = n$, $\dim W = m$. Then $\dim(L(V,W)) = mn$.

Pf:- Follow book (Mapa)

- **Probl:-** Let X, Y, Z be finite dimensional vector spaces over a field F and $T: X \to Y$, $S: Y \to Z$ are linear maps. Prove that rank of $ST \le \min\{\text{rank of } S, \text{rank of } T\}$.

Soln:- $ST: X \to Z$ is a linear map. Now $\operatorname{Im}(ST) \subseteq \operatorname{Im} S$ (prove it) and $\ker T \subseteq \ker(ST)$ (prove it).
$\therefore \dim(\operatorname{Im}(ST)) \le \dim(\operatorname{Im} S)$, i.e., $\operatorname{rank}(ST) \le$ rank of S ——— (1)
and $\dim(\ker T) \le \dim(\ker ST)$ ——— (2)
Now, $\dim(\operatorname{Im} T) + \dim(\ker T) = \dim X = \dim(\operatorname{Im}(ST)) + \dim(\ker(ST))$. ——— (3)
From (2) and (3), we get $\dim(\operatorname{Im}(ST)) \le \dim(\operatorname{Im} T)$.
i.e, rank of $ST \le$ rank of T ——— (4)
From (1) and (4), we get rank $ST \le \min\{\text{rank of } S, \text{rank of } T\}$.

- **Probl:-** Let V and W be finite dimensional vector spaces over a field F and $T, S: V \to W$ are linear maps. Prove that rank of $(S+T) \le$ rank of S + rank of T.

Soln: clearly $Im(S+T) \subseteq ImS + ImT$ (prove it)

$\therefore \dim(Im(S+T)) \leq \dim(ImS + ImT)$

$= \dim(ImS) + \dim(ImT) - \dim(ImS \cap ImT)$ (prove it)

$\leq \dim(ImS) + \dim(ImT)$

i.e, rank of $(S+T) \leq$ rank of S + rank of T.

• Probl:- Let $S, T \in L(V,W)$ where V, W are finite dimensional vector spaces over a field F with $\dim V = n$. Prove that nullity of $(S+T) \geq$ nullity of S + nullity of $T - n$.

Soln:- From the last problem, we have

$n -$ nullity of $(S+T) \leq n -$ nullity of $S + n -$ nullity of T.

$\Rightarrow$ nullity of $(S+T) \geq$ nullity of S + nullity of $T - n$.

• Th:- Let V and W be finite dimensional vector spaces over a field F with $\dim V = n$, $\dim W = m$. Let $T \in L(V,W)$. Let A be the matrix of T relative to a pair of ordered bases of V and W, and B be the matrix of T relative to a different pair of ordered bases of V and W. Then $\exists$ nonsingular matrices P of order $m \times m$ and Q of order $n \times n$ s.t.

$B = P^{-1}AQ$

Pf:- Follow book (Maps)

H.T. probl:- Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be an ordered basis of a real vector space V and a linear map $T: V \to V$ is defined by $T(\alpha_1) = \alpha_1 + \alpha_2 + \alpha_3$, $T(\alpha_2) = \alpha_1 + \alpha_2$, $T(\alpha_3) = \alpha_1$. Show that T is non-singular. Find $m(T^{-1})$ relative to the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$.

• Th:- Let V be a vector space ~~over a field F~~ of dimension n over a field F and let $T \in L(V,V)$. Let ~~$\{\alpha_1, \ldots\}$~~ A be the matrix of T relative to an ordered basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of V and B be the matrix of T relative to an ordered basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ of V. Then $\exists$ a non-singular matrix P of order $n \times n$ ~~s.t.~~ over F s.t.

$B = P^{-1}AP$.

Pf:- Follow ~~book~~ (Maps)

• Probl:- D and T are linear maps on the real vector space P_4 of all real polynomials of degree upto 3, defined by $D(f(x)) = \frac{d}{dx}(f(x))$, $\forall f(x) \in P_4$ and $T(f(x)) = x\frac{d}{dx}(f(x))$, $\forall f(x) \in P_4$. Relative to the ordered basis $\{1, x, x^2, x^3\}$ of P_4, determine the matrix of (i) D (ii) T (iii) $TD - DT$.

Soln:- (i) $D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$,

$D(x) = 1 = 1.1 + 0x + 0x^2 + 0x^3$

$D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$

$D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$

$$\therefore M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly we shall get the other matrices.

• Probl:- Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map which maps the ordered basis $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ of $\mathbb{R}^3$ as: $T(e_1) = (2,3,4)$, $T(e_2) = (1,2,3)$, $T(e_3) = (1,1,1)$. Find $M(T)$ and show that T is not invertible.

Soln:- Find $M(T)$ and show that $\det(M(T)) = 0$ so that $M(T)$ is singular. Hence T is not invertible.

• probl:- A linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (2x+z, x+y+z, -3x-2z)$, $\forall (x,y,z) \in \mathbb{R}^3$.
Show that (i) T is an isomorphism and (ii) $T^{-1} = T$.

Soln:- Consider the ordered basis $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$. Then $T(1,0,0) = (2,1,-3) = 2(1,0,0) + 1.(0,1,0) + (-3)(0,0,1)$

$T(0,1,0) = (0,1,0) = 0.(1,0,0) + 1.(0,1,0) + 0.(0,0,1)$

$T(0,0,1) = (1,1,-2) = 1.(1,0,0) + 1.(0,1,0) + (-2)(0,0,1)$.

$$\therefore M(T) = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ -3 & 0 & -2 \end{pmatrix}.$$

Now $\det(M(T)) = -4 + 3 = -1 \neq 0$ so that $M(T)$ is non-singular.

$\therefore$ T is invertible so that T is bijective.

$\therefore$ T is an isomorphism ——— (i)

$$\text{Now, } (M(T))^{-1} = -\begin{pmatrix} -2 & 0 & -1 \\ -1 & -1 & -1 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ -3 & 0 & -2 \end{pmatrix} = M(T) \therefore = M(T^{-1})$$

$\therefore$ $T = T^{-1}$.

• Probl:- The matrix A of a linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the ordered basis $S = \{(-1,1,1), (1,-1,1), (1,1,-1)\}$ of $\mathbb{R}^3$ is $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. Find the matrix of T relative to the ordered basis $V = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$.

Soln:- Observing A, we have $T(-1,1,1) = (-1,1,1) + 2(1,-1,1) + 3(1,1,-1)$
$= (4,2,0)$

$T(1,-1,1) = 2(-1,1,1) + (1,-1,1) + 3(1,1,-1) = (2,4,0)$,

$T(1,1,-1) = 2(-1,1,1) + 3(1,-1,1) + (1,1,-1) = (2,0,4)$.

For $(x,y,z) \in \mathbb{R}^3$, let $(x,y,z) = c_1(-1,1,1) + c_2(1,-1,1) + c_3(1,1,-1)$

Then, $-c_1+c_2+c_3 = x$ — (1), $c_1-c_2+c_3 = y$ — (2), $c_1+c_2-c_3 = z$ — (3)

From (1), (2), (3), we get $c_1+c_2+c_3 = x+y+z$ — (4)

From (1), (2), (3), (4), we get $c_1 = \frac{y+z}{2}$, $c_2 = \frac{x+z}{2}$, $c_3 = \frac{x+y}{2}$

Now, $\forall (x,y,z) \in \mathbb{R}^3$, we have

$$T(x,y,z) = c_1 T(-1,1,1) + c_2 T(1,-1,1) + c_3 T(1,1,-1)$$
$$= \frac{y+z}{2}(4,2,0) + \frac{x+z}{2}(2,4,0) + \frac{x+y}{2}(2,0,4)$$
$$= (2y+2z+x+z+x+y,\ y+z+2x+2z,\ 2x+2y)$$

$\therefore T(x,y,z) = (2x+3y+3z,\ 2x+y+3z,\ 2x+2y),\ \forall (x,y,z) \in \mathbb{R}^3$.

Now find the matrix of T relative to the ordered basis $\{(1,0,0), (0,1,0), (0,0,1)\}$.

• **Probl:-** Let X, Y, Z be finite dimensional vector spaces over a field F and $T: X \to Y$, $S: Y \to Z$ are linear maps.
Prove that (i) rank of (ST) = rank of S, if T is surjective.
(ii) rank of (ST) = rank of T, if S is injective.

Soln:- (i) We shall show that $Im(ST) = Im(S)$.
Clearly $Im(ST) \subseteq Im(S)$.
Let $z = S(y) \in Im(S)$ be arbitrary. Then $y \in Y$.
Now T is surjective $\Rightarrow \exists x \in X$ s.t. $y = T(x)$.
$\therefore z = S(y) = (ST)(x) \in Im(ST)$.
$\therefore Im S \subseteq Im(ST)$.
$\therefore Im(ST) = Im(S)$
$\Rightarrow \dim(Im(ST)) = \dim(Im(S))$
$\Rightarrow$ rank of (ST) = rank of (S).

$R(T) + N(T) = \dim X$
$= R(ST) + N(ST)$
$R(S) = \dim Y$ ($\because N(S) = 0$)
$R(ST) \leq \min\{R(S), R(T)\}$
$\leq R(S), R(T)$

(ii) Let S be injective.
We shall show that $\ker T = \ker(ST)$.
Clearly, $\ker T \subseteq \ker(ST)$.
Let $x \in \ker(ST)$ be arbitrary. Then $ST(x) = \theta_Z \Rightarrow T(x) = \theta_Y$ ($\because S$ is injective)
$\Rightarrow x \in \ker T$.
$\therefore \ker(ST) \subseteq \ker T$.
$\therefore \ker(T) = \ker(ST)$.
$\therefore \dim(\ker T) = \dim(\ker(ST))$
$\Rightarrow$ nullity of (T) = nullity of (ST) — (1).
Now, rank of T + nullity of T = $\dim X$ = rank of ST + nullity of ST
$\Rightarrow$ rank of ST = rank of T (by (1)).

• **Prob:-** A linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by
$T(x,y,z) = (x+y+z,\ x+z,\ x+y),\ \forall (x,y,z) \in \mathbb{R}^3$.
If A and B be the matrices of T relative to the ordered bases $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3$ respectively, prove that A and B are similar matrices.

• **Defn:-** An $n\times n$ matrix A over a field F is said to be similar to an $n\times n$ matrix B over F if $\exists$ a non-singular matrix P of order $n\times n$ over F s.t. $B = P^{-1}AP$.

• **Note:-** From the above definition, it is clear that if A is similar to B, then B is similar to A. And in this case, we say that A and B are similar matrices.

• **Soln. of the problem:-**
We have $T(1,0,0) = (1,1,1) = (1,0,0)+(0,1,0)+(0,0,1)$

$T(0,1,0) = (1,0,1) = (1,0,0)+0.(0,1,0)+(0,0,1)$

and $T(0,0,1) = (1,1,0) = (1,0,0)+(0,1,0)+0.(0,0,1)$.

$$\therefore A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Again, $T(0,1,1) = (2,1,1) = b_{11}(0,1,1)+b_{21}(1,0,1)+b_{31}(1,1,0)$ (say).

Then, $b_{21}+b_{31}=2,\ b_{11}+b_{31}=1,\ b_{11}+b_{21}=1$.

$\Rightarrow b_{11}+b_{21}+b_{31} = \frac{4}{2} = 2$.

$\therefore b_{11}=0,\ b_{21}=1,\ b_{31}=1$.

$T(1,0,1) = (2,2,1) = b_{12}(0,1,1)+b_{22}(1,0,1)+b_{32}(1,1,0)$ (say).

Then, $b_{22}+b_{32}=2,\ b_{12}+b_{32}=2,\ b_{12}+b_{22}=1$.

$\Rightarrow b_{12}+b_{22}+b_{32} = \frac{5}{2}$

$\therefore b_{12}=\frac{1}{2},\ b_{22}=\frac{1}{2},\ b_{32}=\frac{3}{2}$

$T(1,1,0) = (2,1,2) = b_{13}(0,1,1)+b_{23}(1,0,1)+b_{33}(1,1,0)$ (say).

Then $b_{23}+b_{33}=2,\ b_{13}+b_{33}=1,\ b_{13}+b_{23}=2$.

$\Rightarrow b_{13}+b_{23}+b_{33} = \frac{5}{2}$.

$\therefore b_{13}=\frac{1}{2},\ b_{23}=\frac{3}{2},\ b_{33}=\frac{1}{2}$.

$$\therefore B = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 1/2 & 3/2 \\ 1 & 3/2 & 1/2 \end{pmatrix}$$

Let $S: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map s.t.

$S(1,0,0) = (0,1,1) = 0.(1,0,0)+(0,1,0)+(0,0,1)$

$S(0,1,0) = (1,0,1) = (1,0,0)+0.(0,1,0)+(0,0,1)$

$S(0,0,1) = (1,1,0) = (1,0,0)+(0,1,0)+0.(0,0,1)$.

Then P = matrix of S relative to the ordered basis $\{(1,0,0),(0,1,0),(0,0,1)\} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ which is clearly non-singular, since S maps a basis of $\mathbb{R}^3$ to a basis of $\mathbb{R}^3$ so that S is invertible.

Now, $\forall (x,y,z) \in \mathbb{R}^3$, we have

$$S(x,y,z) = xS(1,0,0) + yS(0,1,0) + zS(0,0,1)$$
$$= x(0,1,1) + y(1,0,1) + z(1,1,0)$$
$$= (y+z, x+z, x+y).$$
$$= (X, Y, Z), \text{ where}$$
$$X = y+z,\ Y = x+z,\ Z = x+y.$$

$$\boxed{x+y+z = \frac{X+Y+Z}{2}}$$

$$\therefore x = \frac{Y+Z-X}{2},\ y = \frac{X-Y+Z}{2},\ z = \frac{X+Y-Z}{2}.$$

$$\therefore S^{-1}(x,y,z) = \left(\frac{y+z-x}{2}, \frac{x-y+z}{2}, \frac{x+y-z}{2}\right),\ \forall (x,y,z) \in \mathbb{R}^3.$$

$$\therefore S^{-1}(1,0,0) = \left(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) = -\tfrac{1}{2}(1,0,0) + \tfrac{1}{2}(0,1,0) + \tfrac{1}{2}(0,0,1),$$
$$S^{-1}(0,1,0) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right) = \tfrac{1}{2}(1,0,0) - \tfrac{1}{2}(0,1,0) + \tfrac{1}{2}(0,0,1)$$
$$S^{-1}(0,0,1) = \left(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right) = \tfrac{1}{2}(1,0,0) + \tfrac{1}{2}(0,1,0) - \tfrac{1}{2}(0,0,1).$$

$$\therefore P^{-1} = m(S^{-1}) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

$$\text{Now, } P^{-1}AP = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} = B.$$

$\therefore$ A and B are similar matrices.

- **Change of coordinate matrix:-**

3.3 Change of coordinate matrix

Let V be a finite dimensional vector space over $\mathbb{R}$ and $\{e_1, e_2, \dots, e_n\}$ a basis of V. Then for a given $v \in V$, there exists unique scalars $a_1, a_2, \dots, a_n$ such that

$v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$

Define a function $x_i : V \to \mathbb{R}$ by $x_i(v) = a_i$ for $i = 1,2,\dots,n$. These functions x_i's are called the **coordinate functions** of v with respect to the basis $\{e_1, e_2, \dots, e_n\}$.

This is the significance of the basis. For a given basis we get a coordinate system on V. Thus we see that n coordinates are required to determine an arbitrary vector $v \in V$ uniquely if $\dim V = n$.

In other words, a vector space V is n −dimensional if it requires n coordinates to locate the points uniquely. In physics, this is said as ' degrees of freedom for a particle $v \in V$ to move'.

For example, let $W = \{(x, y, z) \in \mathbb{R}^3 : y = z\}$. We know that W is a subspace of $\mathbb{R}^3$ but what is the dimension of W? Look, if $(x, y, z) \in W$ then we have, $(x, y, z) = (x, y, y)$. Thus, we need only two quantities x and y to locate a point in W. Thus, we can expect that the dimension of W is 2, in fact, it is.

But what happens if we change the basis of V? That is, if we take another basis of V, definitely, coordinates of v changes. Now we shall discuss it thoroughly.

In geometry, the change of variable

$$x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y'$$

$$y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'$$

can be used to transform the equation $2x^2 - 4xy + 5y^2 = 1$ into the simpler equation $(x')^2 + 6(y')^2 = 1$, in which form it is easily seen to be the equation of an ellipse. Geometrically, the change of variable

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix}$$

is a change in the way that the position of a point P in the plane is described. Thus, to locate a point P having the coordinate $\begin{pmatrix} x \\ y \end{pmatrix}$, we have used the standard basis $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ as

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

After the transformation, the coordinate of P becomes $\begin{pmatrix} x' \\ y' \end{pmatrix}$ where another basis

$\left\{\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}\right\}$ is used as

$$\begin{pmatrix} x \\ y \end{pmatrix} = x'\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} + y'\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

Thus, we have used a new frame of reference, an $x', y'-$ coordinate system with coordinate axes rotated from the original $xy-$ coordinate axes where the new coordinate axes are chosen to lie in the direction of the axes of the ellipse. The unit vectors along the x' $-$axis and the y' $-$axis form an ordered basis

$$B' = \left\{ \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \right\}$$

for $\mathbb{R}^2$ and the change of variable is actually a change from $[P]_B = \begin{pmatrix} x \\ y \end{pmatrix}$, the coordinate vector of P relative to the standard ordered basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, to $[P]'_B = \begin{pmatrix} x' \\ y' \end{pmatrix}$, the coordinate vector of P relative to the new rotated basis B'.

Now, one may ask '*How can a coordinate vector relative to one basis be changed into a coordinate vector relative to the other?*'

Look, the system of equations relating to the new and old coordinates can be represented by the matrix equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

which is in the form $X = QX'$ where

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}, \quad X' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

What is Q?

If $v = \begin{pmatrix} x \\ y \end{pmatrix} \in V = \mathbb{R}^2$ relative to the standard basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and I_V be the identity transformation on V, then

$$I_V(v) = v = x'\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} + y'\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

Thus, the coordinate of v is $\begin{pmatrix} x' \\ y' \end{pmatrix}$ relative to the ordered basis $B' = \left\{ \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \right\}$.

Hence, Q is the matrix representation of I_V, with respect to the ordered bases B and B' respectively and $[v]_B = Q[v]_{B'}$. A similar result is true in general. We state a theorem without proof.

3.2.1 Theorem. Let B and B' be two ordered bases for a finite dimensional vector space V, and let $Q = [I_V]^B_{B'}$. Then

(i) Q is invertible.

(ii) For any $v \in V$, $[v]_B = Q[v]_{B'}$.

The matrix $Q = [I_V]^B_{B'}$, is called a **change of coordinate matrix.**

Example. In $\mathbb{R}^2$, let $B = \{(1,1), (1,-1)\}$ and $B' = \{(2,4), (3,1)\}$ be the ordered bases of $\mathbb{R}^2$.

Now, $(2,4) = 3(1,1) - 1(1,-1)$ and $(3,1) = 2(1,1) + 1(1,-1)$.

The matrix that changes B' –coordinates into B – coordinate is

$$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

But remember one thing, if $B = B'$, i.e. $v_i = w_i$ for $i = 1,2, \ldots, n$, that is, the output basis is the same as input basis, then we have, for identity transformation,

$$I_V(v_1) = 1.v_1 + 0.v_2 + \cdots + 0.v_n$$

$$I_V(v_2) = 0.v_1 + 1.v_2 + \cdots + 0.v_n$$

....

$$I_V(v_n) = 0.v_1 + 0.v_2 + \cdots + 1.v_n$$

Thus the matrix representation of I_V is the identity matrix, I_n.

Hence, the identity transformation is represented by identity matrix only when the output basis and input basis are the same.

If the bases are different, that is, input basis is $\{v_1, v_2, \ldots, v_n\}$ and output basis is $\{w_1, w_2, \ldots, w_n\}$ then we have,

$$I_V(v_j) = \sum_{i=1}^{n} a_{ij} w_i$$

and the change of matrix is $A = (a_{ij})_{n \times n}$.

It is to be noted that the basis is changing but the vectors themselves are not changing : $I_V(v) = v$ for all $v \in V$.

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