-1

... different systems of unit

System of unit is a matter of ~~cot~~ choice. There is no restriction on the particular fundamental and derived units. Also the no. of fundamental unit is a matter of choice. In electrodynamics a variety of unit systems ~~is~~ are in use. It is necessary to understand the changes of basic equations from one system of unit to another.

The dimension of current I in terms of the charge q is given by the equation

$$I = \frac{dq}{dt}$$

The continuity equation is then

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0.$$

The force of interaction between static charges q and q′ is given by Coulomb's law. It says

$$F = K_1 \frac{qq'}{r^2}$$

In c.G.S. electrostatic unit, the unit of charge is defined from this relation by taking $K_1 = 1$. The dimension of charge then comes out to be

$$[q]^2 = ML^3T^{-2}$$

i.e. $[q] = M^{1/2}L^{3/2}T^{-1}$

However there is no hard and fast rule that only M, L and T are to be considered as fundamental unit. In the SI system the unit of current is chosen as the ampere (A) which is that constant current which, if maintained in two straight parallel currents of infinite length and placed one meter apart in vacuum would produce between these conductors a force of 2.10^{-7} newton per unit meter of length. In this system the numerical value of K_1 is $\frac{1}{4\pi\epsilon_0} = 10^{-7}c^2$. If we donot chose any particular system of unit, all we can say is

$$[K_1 q q'] = ML^3T^{-2}$$

The electric field $\vec{E}$ is defined as the force per unit test charge

$$E = K_1 \frac{q}{r^2}$$

The force of interaction between two parallel currents I and I' ~~is~~ of infinite length is given by

$$\frac{\Delta F}{\Delta l} = 2K_2 \frac{II'}{d}$$

where d is the separation between the conductors. Since

$$I = \frac{dq}{dt}, \quad [I] = [q]T^{-1}$$

$$\frac{MLT^{-2}}{L} = [K_2 q q'] \times \frac{T^{-2}}{L}$$

$$\text{or,} \quad [K_2 q q'] = ML$$

$$\left[\frac{K_1}{K_2}\right] = \frac{[K_1 q q']}{[K_2 q q']} = \frac{ML^3T^{-2}}{ML} = L^2T^{-2}$$

Using known charges and currents the numerical value of $\frac{K_1}{K_2} = (3 \times 10^8)^2$. So

$$\frac{K_1}{K_2} = c^2$$

A relation which is to be satisfied by any system of unit. The magnetic induction B is defined through Ampere's law

$$B = 2K_2 \alpha \frac{I}{d}$$

where we have introduced another dimensionful constant α.

$$[E] = [K_1 q] L^{-2}$$

$$[B] = [K_2 \alpha q] T^{-1} L^{-1}$$

So $$\frac{[E]}{[B]} = \left[\frac{K_1}{K_2}\right] \cdot \frac{1}{LT^{-1}\alpha} = \frac{L^2T^{-2}}{LT^{-1}\alpha} = \frac{L}{T\alpha}$$

This relation is useful in writing Faraday's law

$$\vec{\nabla} \times \vec{E} + K_3 \frac{\partial \vec{B}}{\partial t} = 0$$

$$\frac{[E]}{L} = [K_3] \cdot \frac{[B]}{T}$$

$$[K_3] = \frac{[E]}{[B]} \cdot \frac{T}{L} = \frac{L}{T[\alpha]} \cdot \frac{T}{L} = \frac{1}{[\alpha]}$$

Using the foregoing analysis we can write the Maxwell equati[ons] in presence of charges and currents as

$$\vec{\nabla}\cdot\vec{E} = 4\pi K_1 \rho$$

$$\vec{\nabla}\times\vec{B} = 4\pi K_2 \alpha \vec{J} + \frac{K_2 \alpha}{K_1}\frac{\partial \vec{E}}{\partial t}$$

The second term is Maxwell's correction term following from

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

or, $$\vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\left(\frac{1}{4\pi K_1}\frac{\partial \vec{E}}{\partial t}\right) = 0$$

or, $$\vec{\nabla}\cdot\left(\vec{J} + \frac{1}{4\pi K_1}\frac{\partial \vec{E}}{\partial t}\right) = 0$$

and $$\vec{\nabla}\times\vec{B} = 4\pi K_2 \alpha\left(\vec{J} + \frac{1}{4\pi K_1}\frac{\partial \vec{E}}{\partial t}\right)$$

$$= 4\pi K_2 \alpha \vec{J} + \frac{K_2 \alpha}{K_1}\frac{\partial \vec{E}}{\partial t}$$

Also $$\vec{\nabla}\times\vec{E} + K_3 \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

In the source free region

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \frac{K_2 \alpha}{K_1}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

or, $$\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2 \vec{B} = \frac{K_2 \alpha}{K_1} \cdot - K_3 \frac{\partial^2 \vec{B}}{\partial t^2}$$

or, $$\nabla^2 \vec{B} - \frac{K_2 K_3 \alpha}{K_1}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

$$\frac{K_2 K_3 \alpha}{K_1} = \frac{1}{c^2}$$

or, $$K_3 = \frac{1}{\alpha}$$

Thus have the following relations between the constants K_1, K_2, K_3 and α:

$$\frac{K_1}{K_2} = c^2$$

$$K_3 = \frac{1}{\alpha}$$

<u>Different System of units</u>

	K_1	K_2	K_3	α
C.G.S Gaussian	1	$c^{-2}(L^{-2}T^{2})$	c^{-1}	c
S I	$\frac{1}{4\pi\epsilon_0} = 10^{-7}c^2$	$\frac{\mu_0}{4\pi} = 10^{-7}$	1	1

In The S.I system unit of current (A) is taken as The fourth fundamental unit. From

$$\frac{\Delta F}{\Delta l} = 2K_2 \frac{II'}{d}$$

We have from definition of ampere

$$2 \cdot 10^{-7} = 2K_2 \Rightarrow K_2 = 10^{-7}$$

The constant K_2 is written as $K_2 = \frac{\mu_0}{4\pi}$. Then $K_1 = 10^{-7}c^2$. This is written as $K_1 = \frac{1}{4\pi\epsilon_0}$. Also in SI system $\alpha = 1$. This means that $\frac{[E]}{[B]} = LT^{-1}$! The Maxwell's equations are written in SI system as

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

In the Gaussian system $K_1 = 1$. So $K_2 = \frac{1}{c^2}$. Also $\alpha = c$. So $K_3 = \frac{1}{c}$. The same Maxwell equations in this system are

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

The Lorentz force law

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \qquad \text{SI}$$

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) \qquad \text{Gaussian}$$

<u>Exercise</u> Read Appendix on units and dimensions in Jackson.

e-2 (02.07.12)

...ferences

1. Classical Electricity and Magnetism — Panofsky and Phillips

2. Classical Electrodynamics — J. D. Jackson

3. Introduction to Electrodynamics — Griffiths
4. Classical Theory of Electricity and Magnetism — A. K. Raychaudhuri

The Maxwell's equations in vacuum in presence of charges and currents are (in SI units)

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

where ρ is the charge density, $\vec{J}$ is the current denity. The Maxwell's equations automatically imply conservation of charge

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

The electric field $\vec{E}$ and the magnetic induction $\vec{B}$ are defined by the Lorentz force equation

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

In presence of material medium not all charge and current densities can be controlled ~~&~~ from outside. Thus equa

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho + \rho_b}{\epsilon_0}$$

where $\rho_b = -\vec{\nabla}\cdot\vec{P}$ is the bound charge density, $\vec{P}$ being the induced dipole moment per unit volume. We can write

$$\vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P}) = \rho$$

writing $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ (Displacement vector)

$$\vec{\nabla}\cdot\vec{D} = \rho$$

The usefulness of the auxiliary field $\vec{D}$ is that its source is the free charge density only. In linear isotropic dielectric

$$\vec{D} = \epsilon\vec{E} = \epsilon_0 K \vec{E}$$

$K = \epsilon/\epsilon_0$ is the dielectric constant of the medium.

If only linear medium is assumed

$$D_i = \epsilon_0 K_{ij} E_j$$

where Einstein summation convention is adopted.

Similarly, in material medium not all currents can be controlled. Due to magnetization of the medium a 'bound' current density $\vec{J}_b = +\vec{\nabla}\times\vec{M}$ is developed, where $\vec{M}$ = induced magnetic dipole moment per unit volume. Also for time varying polarisation $\vec{P}$ a polarisation current $\frac{\partial\vec{P}}{\partial t}$ is induced. The $\vec{\nabla}\times\vec{B}$

$\vec{D}$ being

equation is thus modified as

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \frac{\partial\vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}\right) + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

i.e. $$\vec{\nabla}\times\frac{\vec{B}}{\mu_0} = \vec{J} + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t} + \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

or, $$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J} + \frac{\partial}{\partial t}(\epsilon_0\vec{E} + \vec{P})$$

Writing $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ as the 'magnetic field intensity'

we get $$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

From definition

$$\vec{B} = \mu_0\vec{H} + \vec{M}$$

For linear isotropic magnetic material, $\vec{M} = \chi_m\mu_0\vec{H}$

$$\vec{B} = \mu\vec{H}, \quad \mu = \mu_0(1+\chi_m)$$

The Maxwell equations in material medium are written as

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

with $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, $\vec{B} = \mu_0\vec{H} + \vec{M}$

In what sense the Maxwell's equations form the basic equations of electromagnetic field system? The electromagnetic field system is constituted of the values of $\vec{E}$ and $\vec{B}$ at every spatial point. The Maxwell equations allow us to obtain the divergence and the curl of $\vec{E}$ and $\vec{B}$. In the static limit these are completely specified by the charge density ρ and the current density $\vec{j}$. There is a general theorem called Helmholtz theorem. The statement of the theorem is as follows:

A vector field is uniquely ~~completely~~ specified in three dimensions if its divergence and curl is given everywhere in space and if these source and circulation densities vanishes sufficiently rapidly at infinity.

We will prove the theorem due to its fundamental importance and also due to the fact that it will help us to fix our notation. Let $\vec{V}(\vec{r})$ be the vector field and

$$\vec{\nabla}\cdot\vec{V}(\vec{r}) = s$$

and

$$\vec{\nabla}\times\vec{V} = \vec{C}$$

We assume s and $\vec{C}$ are given fn.s of coordinates. For consistency $\vec{\nabla}\cdot\vec{C} = 0$. We also assume that s and c goes to zero sufficiently rapidly at infinity. Now

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}\times\vec{C}$$

i.e. $\vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2\vec{V} = \vec{\nabla}\times\vec{C}$

$$\begin{array}{r} 10500 \\ 4750 \\ \hline 15250 \end{array}$$

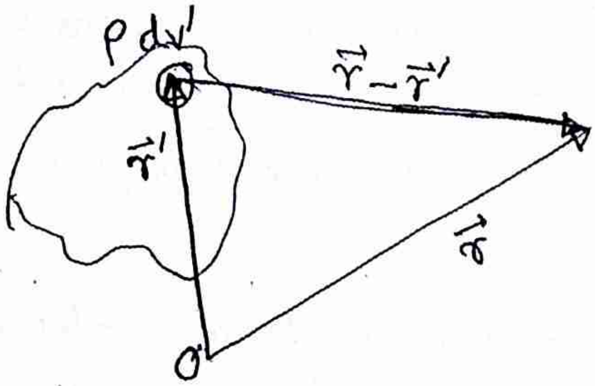
$$\nabla^2 \vec{V} = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \vec{\nabla}\times\vec{C}$$

The components of $\vec{V}$ satisfy the Poisson equation

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

The solution of which is known to be

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dv'$$



Here $\vec{r}\,' \equiv (x', y', z')$ denote the source coordinates and $\vec{r} \equiv (x, y, z)$ denote the field coordinates. The operator $\vec{\nabla}$ will denote gradient with respect to the field coordinates, $\vec{\nabla}'$ that w.r.t. the source coordinates.

The solution to $\vec{V}$ can be written straight forwardly as

$$\vec{V} = -\frac{1}{4\pi}\int \frac{\vec{\nabla}'(\vec{\nabla}'\cdot\vec{V}) - \vec{\nabla}'\times\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dv'$$

$$= -\frac{1}{4\pi}\int \frac{\vec{\nabla}' \rho}{|\vec{r}-\vec{r}'|} dv' + \frac{1}{4\pi}\int \frac{\vec{\nabla}' \times \vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|} dv'$$

The first term $= -\frac{1}{4\pi}\int \vec{\nabla}'\left(\frac{\rho}{|\vec{r}-\vec{r}'|}\right) dv'$

$$+\frac{1}{4\pi}\int \rho(\vec{r}')\, \vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) dv'$$

Now

$$\vec{\nabla}'\left(f(|\vec{r}-\vec{r}'|)\right)$$

$$\equiv \frac{\partial}{\partial x'}\left(f\left(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)\right)$$

$$= \frac{\partial}{\partial x'}\left(f(X, Y, Z)\right) \text{ where } X = x-x', Y = y-y', Z = z-z'$$

$$= \frac{\partial f}{\partial X}\frac{dX}{dx'} = -\frac{\partial f}{\partial X} = \frac{-\partial}{\partial x}\left(f(X,Y,Z)\right) \equiv -\vec{\nabla}\left(f(|\vec{r}-\vec{r}'|)\right)$$

So for action on any function of $|\vec{r}-\vec{r}'|$,

$$\vec{\nabla}' = -\vec{\nabla}$$

Hence the first term

$$= -\frac{1}{4\pi}\oint \frac{\rho}{|\vec{r}-\vec{r}'|} d\vec{s} - \frac{1}{4\pi}\int \rho(\vec{r}')\, \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) dv'$$

$$= -\frac{1}{4\pi}\vec{\nabla}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dv'$$

$$= -\vec{\nabla}\phi \text{ where } \phi(\vec{r}) = \frac{1}{4\pi}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dv'$$

Similarly, the second term

$$\vec{\nabla}\times(\phi\vec{a}) = \phi\vec{\nabla}\times\vec{a} - \vec{a}\times\vec{\nabla}\phi$$

$$= \frac{1}{4\pi}\int \frac{\vec{\nabla}'\times\vec{C}}{|\vec{r}-\vec{r}'|}\,dv'$$

$$= \frac{1}{4\pi}\int\left[\vec{\nabla}'\left(\frac{\vec{C}}{|\vec{r}-\vec{r}'|}\right) + \vec{C}\times\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]dv'$$

$$= -\frac{1}{4\pi}\int \vec{C}\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dv'$$

$$= \frac{1}{4\pi}\int \vec{\nabla}\times\left(\frac{\vec{C}}{|\vec{r}-\vec{r}'|}\right)dv'$$

$$= \vec{\nabla}\times\vec{A}$$

Hence the vector field $\vec{V}$ is expressed as

$$\vec{V} = -\vec{\nabla}\phi + \vec{\nabla}\times\vec{A}$$

We next show that the vector field thus constructed is unique. If not, suppose there are two vector fields $\vec{V}_1$ and $\vec{V}_2$ such that

$$\vec{\nabla}\cdot\vec{V}_1 = s \qquad \vec{\nabla}\times\vec{V}_1 = \vec{C}$$

$$\vec{\nabla}\cdot\vec{V}_2 = s \qquad \vec{\nabla}\times\vec{V}_2 = \vec{C}$$

Let $\vec{W} = \vec{V_1} - \vec{V_2}$. Then

$$\vec{\nabla} \times \vec{W} = 0$$

i.e. $\vec{W} = \vec{\nabla} \Psi$.

But $\vec{\nabla} \cdot \vec{W} = 0$. So $\nabla^2 \Psi = 0$

We require Ψ to be regular everywhere. But the only solution of the laplace equation which is regular everywhere is $\Psi = 0$.

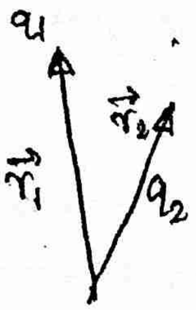
So $\vec{W} = 0$ and $\vec{V_1} = \vec{V_2}$.

The above theorem emphasizes that both divergence and curl of a vectorfield are required to be known to specify a vectorfield uniquely. The Maxwell's equations provide this information. Since for electromagnetic fields $\vec{E}$ & $\vec{B}$, the curl is contributed by the rate of change of fields, the solution is difficult.

Lecture – 3 (03.07.12)

Interaction energy of a system of point charges:

A system comprising of n point charges $q_1, q_2, \dots, q_n$ at $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$



The system is assumed to be built by bringing the charges one by one from infinity to their present locations. No work is required to bring q_1. The work required to bring q_2 after q_1 is

$$W_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

In this way the work required to assemble the system is

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|} + \frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|} + \cdots\right].$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

If we remove the restriction $j > i$ then terms such as $\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$ are counted twice. So

$$W = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \sum_{i=1}^{n} {\sum_{j=1}^{n}}' \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

where the prime over the summation symbol means $j = i$ is not included. Now

$$W = \frac{1}{2} \sum_i q_i \left({\sum_j}' \frac{q_j}{|\vec{r}_i - \vec{r}_j|} \right)$$

$$= \frac{1}{2} \sum_i q_i \phi_i$$

where ϕ_i is the potential at $\vec{r}_i$ due to all charges of the system except the i-th charge.

<u>Field energy</u>:

$$W = \frac{1}{2} \int \rho \phi \, dV \quad \text{for continuous distribution}$$

From Maxwell's equations

$$W = \frac{1}{2} \int (\epsilon_0 \vec{\nabla} \cdot \vec{E}) \phi \, dv$$

$$= \frac{\epsilon_0}{2} \left[\int \vec{\nabla} \cdot (\vec{E} \phi) \, dv - \int \vec{E} \cdot \vec{\nabla} \phi \, dv \right]$$

The first term is reduced to a surface integral over the surface at infinity. ~~The second term~~ Since $E \sim \frac{1}{r^2}$, $\phi \sim \frac{1}{r}$, $ds \sim r^2$ the integral vanishes at infinity.

$$W = \frac{\epsilon_0}{2}\int E^2 dV$$

In this picture energy is distributed throughout with density $\frac{\epsilon_0 E^2}{2}$.

There is one puzzling question. The energy of the n- point charge system $W = \frac{1}{2}\sum_i q_i \phi_i$ may be +ve or -ve, whereas $W = \frac{\epsilon_0}{2}\int E^2 dV$ is +ve definite. To see the reason

$$\vec{E} = \sum_i \vec{E}_i$$

at any point $\vec{r}$. Then

$$E^2 = \sum_i \sum_j \vec{E}_i \cdot \vec{E}_j$$

$$= \sum_i E_i^2 + \sum_i \sum_j{}' \vec{E}_i \cdot \vec{E}_j$$

Then $$W = \sum_i \frac{\epsilon_0}{2}\int E_i^2 dV + \sum_i \sum_j{}' \frac{\epsilon_0}{2}\int \vec{E}_i \cdot \vec{E}_j \, dV$$

The second term is

$$W_{int} = \sum_i \sum_j{}' \frac{\epsilon_0}{2}\int \vec{E}_i \cdot \vec{\nabla}\phi_j \, dV$$

$$= -\sum_i \frac{\epsilon_0}{2}\int \vec{E}_i \cdot \vec{\nabla}\phi \, dV$$

where $\phi = \sum_j{}' \phi_j$ is the potential at $\vec{r}$ due to all except the i-th charge.

$$W_{int} = -\sum_i \frac{\epsilon_0}{2} \int \left[\vec{\nabla}\cdot(\vec{E}_i \phi) - \phi \vec{\nabla}\cdot\vec{E}_i\right] dV$$

The first term vanishes (being a surface term at infinity). In the second term we put $\vec{\nabla}\cdot\vec{E}_i = \frac{q_i}{\epsilon_0}\delta(\vec{r}-\vec{r}_i)$. Then

$$W_{int} = \sum_i \frac{\epsilon_0}{2} \int \phi(\vec{r}) \cdot \frac{q_i}{\epsilon_0} \delta(\vec{r}-\vec{r}_i)\, dV$$

$$= \frac{1}{2}\sum_i q_i \phi_i$$

We thus find that the second term of $W = \frac{\epsilon_0}{2}\int E^2 dV$ is precisely the interaction energy of the n-charge system. The remaining term

$$\sum_i \frac{\epsilon_0}{2} \int E_i^2 dV$$

is identified with self-energy of the system. The self energy ~~of the~~ is actually infinite but that does not bother us because the point charges are given and we can only manipulate the interaction term. This additional term sneaked in our equation

$$\frac{1}{2}\sum_i q_i \phi_i \rightarrow \frac{1}{2}\int \rho \phi\, dV$$

i.e. during the replacement of the discrete system by continuous charge distribution.

[E]lectrostatic energy in dielectric medium :

We consider a virtual process in which free charge density is changed everywhere by $\delta\rho$ with rigid boundaries

$$\delta W = \int \delta\rho \, \phi \, dV$$

Since $\vec{\nabla}\cdot\vec{D} = \rho$ (ρ = free charge density)

$$\delta W = \int \delta(\vec{\nabla}\cdot\vec{D})\,\phi\, dV$$

$$= \int (\vec{\nabla}\cdot\delta\vec{D})\,\phi\, dV$$

$$= \int [\vec{\nabla}\cdot(\phi\,\delta\vec{D}) - \delta\vec{D}\cdot\vec{\nabla}\phi]\, dV$$

The first term on the rhs drops out and

$$\delta W = \int \vec{E}\cdot\delta\vec{D}\, dV$$

Summing over all the δ-variations leading to the build up of the displacement vector from 0 to $\vec{D}$ we will get the energy of the system. For this we require a constitutive reln. between $\vec{D}$ and $\vec{E}$. Assuming linear isotropic dielectric

$$\vec{D} = \epsilon\vec{E} = \epsilon_0 K\vec{E}$$

$$\delta W = \int \vec{E} \cdot \delta(\epsilon_0 k \vec{E})\, dv$$

$$= \int \epsilon_0 k\, \delta\left(\frac{E^2}{2}\right) dv = \frac{1}{2}\delta \int \epsilon_0 k E^2 dv$$

$$= \frac{1}{2}\delta \int \vec{E} \cdot \vec{D}\, dv$$

So $$W = \int_0^{\vec{D}} \delta W = \frac{1}{2}\int \vec{E} \cdot \vec{D}\, dv$$

Infact, only the linearity is important. Thus if

$$D_i = \epsilon_0 k_{ij} E_j$$ (Einstein summation convention assumed).

where k_{ij} is the dielectric tensor

$$\delta W = \int E_K\, \delta D_K\, dv$$

$$= \int E_K\, \delta(\epsilon_0 k_{Kj} E_j)\, dv$$

~~where~~ Now

$$\int E_K\, \delta(\epsilon_0 k_{Kj} E_j)\, dv = \int E_K\, \epsilon_0 k_{Kj} \delta E_j\, dv$$

$$= \int \epsilon_0 k_{Kj} \left[\delta(E_K E_j) - E_j \delta E_K\right] dv$$

$$= \delta \int \epsilon_0 k_{Kj} E_K E_j\, dv - \int \epsilon_0 k_{jK} E_K \delta E_j\, dv$$

$$\Rightarrow \int E_K\, \delta(\epsilon_0 k_{Kj} E_j)\, dv = \frac{1}{2}\delta \int \epsilon_0 k_{Kj} E_K E_j\, dv$$

where $k_{Kj} = k_{jK}$ assumed. $$= \frac{1}{2}\delta \int \vec{E} \cdot \vec{D}\, dv$$

Lecture - 4 (09.07.12)

The interaction of two charges is viewed as the response of the second charge in the field of the first charge in field theory. The field is thus the transmitting media of the forces. A stress is developed due to the field which carries the interaction. We define the Maxwell stress tensor T_{ij} by

$$\boxed{dF_i = T_{ij}\, dS_j}$$

where $d\vec{S}$ is an infinitesimal surface element kept at the point where T_{ij} is determined. Expression for T_{ij} is determined from

$$F_i = \int \frac{\partial T_{ij}}{\partial x_j}\, dV = \int F_i^V\, dV$$

where F_i^V is the i-th component of the force per unit volume. Equating

$$\boxed{\frac{\partial T_{ij}}{\partial x_j} = F_i^V}$$

If we consider a closed surface enclosing no charge

$$\oint T_{ij}\, dS_j = 0$$

where integration is ~~abov~~ over the closed surface.

Expression for T_{ij} in absence of dielectric:

$$\vec{F}^{v} = \rho\vec{E}$$

$$F_i^{v} = \rho E_i = \epsilon_0 (\vec{\nabla}\cdot\vec{E}) E_i$$

$$= \epsilon_0 \frac{\partial E_j}{\partial x_j} E_i = \epsilon_0 \frac{\partial}{\partial x_j}(E_j E_i) - \epsilon_0 E_j \frac{\partial E_i}{\partial x_j}$$

$$= \epsilon_0 \frac{\partial}{\partial x_j}(E_i E_j) - \epsilon_0 E_j \frac{\partial E_j}{\partial x_i}$$

$$E_j = \epsilon_{ipk} \frac{\partial E_k}{\partial x_p}$$

$$\left[\text{As } \vec{\nabla}\times\vec{E} = 0,\ \frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}\right]$$

$$\frac{\partial E_i}{\partial x_j} = \epsilon_{ipk} \frac{\partial^2 E_k}{\partial x_p \partial x_j}$$

$$\epsilon_{jpk} \frac{\partial E_k}{\partial x_p \partial x_i}$$

$$F_i^{v} = \epsilon_0 \frac{\partial}{\partial x_j}(E_i E_j) - \frac{\epsilon_0}{2}\frac{\partial}{\partial x_i}(E_j E_j)$$

$$= \epsilon_0 \left[\frac{\partial}{\partial x_j}\left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right)\right]$$

So $$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right)$$

is the expression for the Maxwell stress tensor in absence of any material medium. In expanded form

$$T_{ij} \equiv \epsilon_0 \begin{pmatrix} \frac{1}{2}(E_x^2 - E_y^2 - E_z^2) & E_x E_y & E_x E_z \\ E_x E_y & \frac{1}{2}(E_y^2 - E_z^2 - E_x^2) & E_y E_z \\ E_x E_z & E_y E_z & \frac{1}{2}(E_z^2 - E_x^2 - E_y^2) \end{pmatrix}$$

x-axis is taken along $\vec{E}$, $E_x = E$, $E_y = E_z = 0$ and

$$T_{ij} \equiv \begin{pmatrix} \frac{\epsilon_0 E^2}{2} & 0 & 0 \\ 0 & -\frac{\epsilon_0 E^2}{2} & 0 \\ 0 & 0 & -\frac{\epsilon_0 E^2}{2} \end{pmatrix}$$

If $d\vec{s}$ is chosen $\parallel^{l}$ to the x-axis, the force on it is

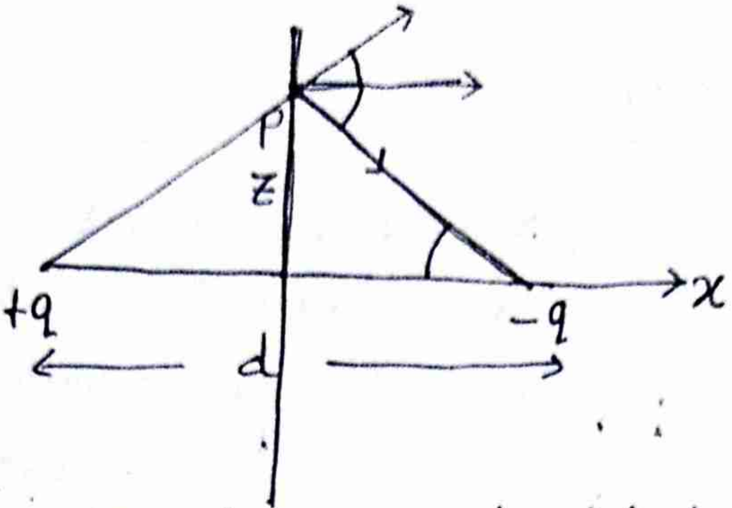
$$F \equiv \begin{pmatrix} \frac{\epsilon_0 E^2}{2} & 0 & 0 \\ 0 & -\frac{\epsilon_0 E^2}{2} & 0 \\ 0 & 0 & -\frac{\epsilon_0 E^2}{2} \end{pmatrix} \begin{pmatrix} dS \\ 0 \\ 0 \end{pmatrix}$$

$F_x = \frac{\epsilon_0 E^2}{2} dS$ is the only nonzero force component. If $d\vec{s}$ is along the y-axis

$$F = F_y = -\frac{\epsilon_0 E^2}{2} dS$$

So there is a tension $\frac{\epsilon_0 E^2}{2}$ along the field and pressure $\frac{\epsilon_0 E^2}{2}$ perpendicular to the field.

Force between two equal and opposite charges q and $-q$ separated by a distance d.



Two charges +q and −q are separated at a distance d in vacuum. To find the force on −q due to +q we encircle −q by any closed surface and find the force exerted across this volume. For convenience we take the infinite half space containing −q. The electric field E at P is

$$E = 2 \cdot \frac{1}{4\pi\epsilon_0} \frac{q}{\left(\frac{d^2}{4} + z^2\right)} \frac{d/2}{\sqrt{\frac{d^2}{4} + z^2}}$$

$$= \frac{qd}{4\pi\epsilon_0 \left(\frac{d^2}{4} + z^2\right)^{3/2}}$$

The force along the x-axis given by

$$-\int_0^\infty \frac{\epsilon_0}{2} \frac{q^2 d^2}{16\pi^2 \epsilon_0^2 \left(\frac{d^2}{4} + z^2\right)^3} \cdot 2\pi z \, dz$$

$$= -\frac{q^2 d^2}{16\pi\epsilon_0} \int_0^\infty \frac{z\,dz}{\left(\frac{d^2}{4} + z^2\right)^3}$$

$$\int_0^\infty \frac{z\,dz}{\left(\frac{d^2}{4}+z^2\right)^3}$$

$$= \int_0^\infty \frac{p\,dp}{p^6} = \int_0^\infty \frac{dp}{p^5} = \left[-\frac{1}{4p^4}\right]_0^\infty$$

$$= \frac{1}{4 \cdot \frac{d^4}{16}} = \frac{4}{d^4}$$

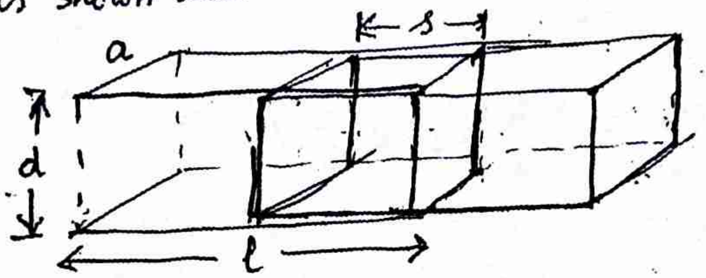
The force $F = -\dfrac{q^2 d^2}{16\pi\epsilon_0} \cdot \dfrac{4}{d^4} = -\dfrac{1}{4\pi\epsilon_0} \dfrac{q^2}{d^2}$. The

$-$ sign shows that it is an attraction.

Lecture-5 (10.07.12)

Force on dielectric

~~If a dielectric medium is in an inhomogeneous external electric field~~ If a dielectric medium is in a state of electrification volume forces act on it. We first consider an example — a dielectric slab is partially inserted within a parallel plate capacitor as shown below



within the capacitor plates the electric field is uniform and the force on the bound charges in the medium is balanced. But at the edge there is nonuniformity. This is the reason behind the force on the dielectric.

To compute the force suppose the dielectric slab is given a virtual displacement δs against the force. The change in energy

$$\delta W = -F\delta s$$

or $$F = -\frac{dW}{ds}$$

But $$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

$$F = +\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{ds} = \frac{1}{2}V^2\frac{dC}{ds}$$

$$C = \frac{\epsilon_0 (l - s) a}{d} + \frac{\epsilon_0 (1 + \chi_e) s a}{d}$$

$$= \frac{\epsilon_0 a}{d} (l + \chi_e s)$$

$$\frac{dC}{ds} = \frac{\epsilon_0 \chi_e a}{d}$$

$$F = \frac{1}{2} V^2 \frac{dC}{ds} = \frac{1}{2} \epsilon_0 \chi_e \frac{a}{d} V^2$$

<u>volume forces in electrostatic field in dielectric medium:</u>

If $\vec{F_v}$ is the volume force (force per unit volume) then in an arbitrary virtual displacement $\delta \vec{x}$

$$\delta W = -\int \vec{F_v} \cdot \delta \vec{r} \, dv$$

So $$\frac{dW}{dt} = -\int \vec{F_v} \cdot \vec{u} \, dv$$

where $\vec{u}$ is an arbitrary velocity field in the dielectric.

Now $$W = \frac{1}{2} \int \vec{E} \cdot \vec{D} \, dv$$

$$\delta W = \frac{1}{2} \delta \int \vec{E} \cdot \vec{D} \, dv$$

Under the type of variation considered both free charge density varies along with variation of dielectric constant.

$$\delta W = \frac{1}{2\epsilon_0}\int D^2 \delta\left(\frac{1}{k}\right) dv + \int \vec{E}\cdot\delta\vec{D}\, dv$$

$$\int \vec{E}\cdot\delta\vec{D}\, dv = -\int \vec{\nabla}\phi\cdot\delta\vec{D}\, dv$$

$$= -\int\left[\vec{\nabla}\cdot(\phi\delta\vec{D}) - \phi\,\delta(\vec{\nabla}\cdot\vec{D})\right] dv$$

$$= \int \phi\,\delta\rho\, dv$$

$$\delta W = \frac{1}{2\epsilon_0}\int -\frac{D^2}{k^2}\delta k\, dv + \int \phi\,\delta\rho\, dv$$

$$\frac{dW}{dt} = \int\left(\phi\frac{\partial\rho}{\partial t} - \frac{\epsilon_0}{2}E^2\frac{\partial k}{\partial t}\right) dv$$

We have the continuity equations following from charge conservation and mass conservation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\frac{\partial\rho_m}{\partial t} + \vec{\nabla}\cdot(\rho_m\vec{u}) = 0$$

The rate of change of the dielectric constant k with time is contributed by both the variation in time and due to the fact that the medium is moving. This is given by the convective derivative

$$\frac{Dk}{Dt} = \frac{\partial k}{\partial t} + \vec{u}\cdot\vec{\nabla}k$$

Also $$\frac{Dk}{Dt} = \frac{dk}{d\rho_m}\frac{D\rho_m}{Dt} = \frac{dk}{d\rho_m}\left(\frac{\partial \rho_m}{\partial t} + \vec{u}\cdot\vec{\nabla}\rho_m\right)$$

$$= \frac{dk}{d\rho_m}\left(-\vec{\nabla}\cdot(\rho_m\vec{u}) + \vec{u}\cdot\vec{\nabla}\rho_m\right)$$

$$= -\frac{dk}{d\rho_m}\rho_m(\vec{\nabla}\cdot\vec{u})$$

So $$\frac{\partial k}{\partial t} + \vec{u}\cdot\vec{\nabla}k = -\frac{dk}{d\rho_m}\rho_m(\vec{\nabla}\cdot\vec{u})$$

i.e. $$\frac{\partial k}{\partial t} = -\frac{dk}{d\rho_m}\rho_m(\vec{\nabla}\cdot\vec{u}) - \vec{u}\cdot\vec{\nabla}k$$

$$\frac{dW}{dt} = \int\Big[-\phi\,\vec{\nabla}\cdot(\rho\vec{u}) + \frac{\epsilon_0}{2}E^2\frac{dk}{d\rho_m}\rho_m(\vec{\nabla}\cdot\vec{u})$$
$$+\left(\frac{\epsilon_0}{2}E^2\nabla k\right)\cdot\vec{u}\Big]dV$$

$$= \int\Big[-\vec{\nabla}\cdot(\rho\phi\vec{u}) + \rho\vec{u}\cdot\vec{\nabla}\phi$$
$$+\vec{\nabla}\cdot\left(\frac{\epsilon_0}{2}E^2\frac{dk}{d\rho_m}\rho_m\vec{u}\right)$$
$$-\vec{\nabla}\left(\frac{\epsilon_0}{2}E^2\frac{dk}{d\rho_m}\rho_m\right)\cdot\vec{u}$$
$$+\left(\frac{\epsilon_0}{2}E^2\nabla k\right)\cdot\vec{u}\Big]dV$$

The first and the Third term are converted to boundary
and vanishes by taking the surface outside the dielectric. Then

$$\frac{dW}{dt} = \int \left[-\rho\vec{E} - \vec{\nabla}\left(\frac{\epsilon_0}{2} E^2 \frac{dk}{d\rho_m}\rho_m\right) + \frac{\epsilon_0}{2} E^2 \vec{\nabla} k \right] \cdot \vec{u}\, dV$$

Comparision gives

$$\vec{F}_v = \rho\vec{E} - \frac{\epsilon_0}{2} E^2 \vec{\nabla} k + \vec{\nabla}\left(\frac{\epsilon_0}{2} E^2 \frac{dk}{d\rho_m}\rho_m\right)$$

The first term is the force on the free charge density. The second term appears when an inhomoginous dielectric is placed in an electric field. The last term vanishes when integrated over the entire dielectric. This is known as the electrostriction term.

Expression of Maxwell's stress tensor ~~in~~ for electrostatic field in dielectric medium:

$$F_i^V = \rho E_i - \frac{\epsilon_0}{2} E^2 \frac{\partial k}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{\epsilon_0}{2} E^2 \frac{dk}{d\rho_m}\rho_m\right)$$

$$\cancel{= \frac{\partial}{\partial x_j}\left(\epsilon_0 E_i E_j - \frac{1}{2}\epsilon_0 E^2 \delta_{ij}\right) - \frac{\epsilon_0}{2} E^2 \frac{\partial k}{\partial x_i}}$$

$$\cancel{+ \frac{\partial}{\partial x_i}\left(\frac{\epsilon_0}{2} E^2 \frac{dk}{d\rho_m}\rho_m\right.}$$

$$\vec{a} = \vec{K}\phi$$

$$\int \phi\, d\vec{S} = \int \nabla\phi\, dv$$

first term

$$\rho E_i = (\vec{\nabla}\cdot\vec{D})E_i$$

$$= \vec{\nabla}\cdot(\epsilon_0 k \vec{E})E_i$$

$$= \frac{\partial}{\partial x_j}(\epsilon_0 k E_j)E_i$$

$$= \epsilon_0 k \frac{\partial E_j}{\partial x_j}E_i + \epsilon_0 \frac{\partial k}{\partial x_j}E_j E_i$$

$$= \cancel{\epsilon_0 k}\ \frac{\partial}{\partial x_j}(\epsilon_0 k E_j E_i) - \epsilon_0 k E_j \frac{\partial E_i}{\partial x_j}$$

$$= \frac{\partial}{\partial x_j}(\epsilon_0 k E_j E_i) - \epsilon_0 k E_j \frac{\partial E_j}{\partial x_i} \quad [\text{as } \vec{\nabla}\times\vec{E}=0]$$

$$= \frac{\partial}{\partial x_j}(\epsilon_0 k E_i E_j) - \epsilon_0 k \frac{\partial}{\partial x_i}\left(\frac{1}{2}E^2\right)$$

$$= \frac{\partial}{\partial x_j}(\epsilon_0 k E_i E_j) - \frac{\partial}{\partial x_i}\left(\epsilon_0 k \cdot \frac{1}{2}E^2\right) + \frac{\epsilon_0}{2}E^2\frac{\partial k}{\partial x_i}$$

$$= \frac{\partial}{\partial x_j}\left(\epsilon_0 k E_i E_j - \frac{1}{2}\epsilon_0 k E^2 \delta_{ij}\right) + \frac{\epsilon_0}{2}E^2\frac{\partial k}{\partial x_i}$$

So $$F_i^v = \frac{\partial}{\partial x_j}\left[E_i D_j - \frac{1}{2}\delta_{ij}(\vec{E}\cdot\vec{D}) + \delta_{ij}\frac{\epsilon_0}{2}E^2\frac{dk}{d\rho_m}\rho_m\right]$$

$$T_{ij} = E_i D_j - \frac{1}{2}\delta_{ij}(\vec{E}\cdot\vec{D})\left(1 - \frac{\rho_m}{k}\frac{dk}{d\rho_m}\right)$$

Lecture – 5 (16.07.12)

Energy of magnetostatic field: Magnetic forces do no work. But in building up a current system work is required to be done against the back e.m.f induced by changing magnetic field.

Consider a closed loop in which I is the current driven by e.m.f. $\mathcal{E}$. The work done per unit time is $\mathcal{E} I = I\int \vec{E}'\cdot d\vec{l}$ where $\vec{E}'$ is the electric field produced by the source. For an arbitrary current system the generalisation is obvious. The work done by the sources producing e.m.f. is given by (per unit time)

$$\frac{dW}{dt} = \int \vec{E}'\cdot\vec{j}\, dV .$$

Now $\vec{j} = \sigma(\vec{E} + \vec{E}')$ i.e. $\frac{\vec{j}}{\sigma} = \vec{E}' + \vec{E}$

$$\int \vec{E}'\cdot\vec{j}\, dV = \int \frac{j^2}{\sigma} dV - \int \vec{E}\cdot\vec{j}\, dV$$

The first term on the r.h.s. is the Joule heat loss. The second term is then the energy fed into the magnetic field. If U_m represents magnetic field energy

$$\frac{dU_m}{dt} = -\int \vec{E}\cdot\vec{j}\, dv$$

$$R\cdot H.S = -\int \vec{E}\cdot \vec{\nabla}\times\vec{H}\, dv$$

$$= \int\left[\vec{\nabla}\cdot(\vec{E}\times\vec{H}) - \vec{H}\cdot\vec{\nabla}\times\vec{E}\right]dv$$

For static field the first term drops out and

$$\frac{dU_m}{dt} = +\int \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\, dv$$

Thus if we consider a virtual process in which the current system is increased by an infinitesmal amount so that change in induction is $\delta\vec{B}$, the change in U_m is

$$\delta U_m = \int \vec{H}\cdot\delta\vec{B}\, dv$$

Assuming a linear relation between $\vec{B}$ and $\vec{H}$

$$U_m = \frac{1}{2}\int \vec{H}\cdot\vec{B}\, dv$$

Note that the particular process is assumed to be quasistatic so that the displacement current is negligable w.r.t. the conduction current which enabled us to take

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

The expression for U_m can be written in terms of the sources: Put $\vec{B} = \vec{\nabla} \times \vec{A}$ and

$$U_m = \frac{1}{2} \int \vec{H} \cdot \vec{\nabla} \times \vec{A} \, dv$$

$$= \frac{1}{2} \int \left[\vec{\nabla} \cdot \left(\vec{A} \times \vec{H} \right) + \vec{A} \cdot \left(\vec{\nabla} \times \vec{H} \right) \right] dv$$

The first term again drops out and

$$U_m = \frac{1}{2} \int \vec{A} \cdot \vec{J} \, dv$$

Note that this relation holds true for a linear medium only.

Maxwell's stress tensor for magnetostatic field

For simplification we will consider magnetostatic field in vacuo due to specified source distribution. The volume force

$$\vec{F}^v = \vec{J} \times \vec{B}$$

The Maxwell's stress tensor is once again defined by

$$\frac{\partial T_{ij}}{\partial x_j} = F_i^v$$

One can write the cross-product between two vectors $\vec{a} \times \vec{b}$ as

$$(\vec{a} \times \vec{b})_i = \epsilon_{ikl} a_k b_l$$

where ϵ_{ikl} is the Levi Civita symbol.

$$F_i^v = \epsilon_{ikm} j_k B_m$$

i.e. $$= \epsilon_{ikm} (\vec{\nabla} \times \vec{H})_k B_m \quad [\vec{B} = \mu_0 \vec{H}].$$

$$= \epsilon_{ikm} \cdot \frac{1}{\mu_0} \epsilon_{krs} \frac{\partial B_s}{\partial x^r} B_m$$

$$= \frac{-1}{\mu_0} \epsilon_{kim} \epsilon_{krs} \frac{\partial B_s}{\partial x^r} B_m$$

Now $\epsilon_{kim}\epsilon_{krs} = \delta_{ir}\delta_{ms} - \delta_{is}\delta_{mr}$

So the last expression

$$= \frac{1}{\mu_0}(\delta_{is}\delta_{mr} - \delta_{ir}\delta_{ms}) \frac{\partial B_s}{\partial x^r} B_m$$

$$= \frac{1}{\mu_0}\left(\frac{\partial B_i}{\partial x^m} B_m - \frac{\partial B_m}{\partial x^i} B_m\right)$$

$$= \frac{1}{\mu_0}\left[\frac{\partial}{\partial x^m}(B_i B_m) - B_i \frac{\partial B_m}{\partial x_m} - \frac{\partial}{\partial x^i}\left(\frac{1}{2}B^2\right)\right]$$

But $\frac{\partial B_m}{\partial x_m} = \vec{\nabla} \cdot \vec{B} = 0$. Hence

$$\frac{\partial T_{im}}{\partial x_m} = \frac{1}{\mu_0} \frac{\partial}{\partial x_m}\left[B_i B_m - \frac{1}{2}\delta_{im} B^2\right]$$

Hence $T_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$

Formally, one can write

$$T_{ij} = \left(H_i B_j - \frac{1}{2}\delta_{ij}\, \vec{H}\cdot\vec{B}\right)$$

The expression agrees with Maxwell's stress tensor for magnetostatic field in material medium under the assumptions that 1) the medium is linear i.e. its permeability is not a function of the field 2) There is no permanent magnetic moment present and 3) There is no magnetostriction $\left(\frac{dK_m}{d\rho_m} = 0\right)$.

$$\vec{J}\times\vec{B} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$$

$$\vec{\nabla}\times(\vec{a}\times\vec{b}) = \cancel{(\vec{b}\cdot\vec{\nabla})\vec{a}}\,(\vec{\nabla}_a\times\vec{a})\times\vec{b}$$

$$- \cancel{\oplus}\,(\cancel{\nabla}_b \quad \vec{a}\times(\vec{\nabla}\times\vec{b})$$

$$\vec{\nabla}(\vec{a}\cdot\vec{b}) = \vec{a}\times(\vec{\nabla}\times\vec{b}) + \vec{b}\times(\vec{\nabla}\times\vec{a})$$

$$+ (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a}$$

Here $\vec{a} = \vec{b} = \vec{B}$

$$2(\vec{\nabla}\times\vec{B})\times\vec{B} = -\vec{\nabla}(B^2)$$

$$- 2(\vec{B}\cdot\vec{\nabla})\vec{B}$$

$$(\vec{J}\times\vec{B})_K = -\frac{1}{2}\frac{\partial}{\partial x^K}(B^2) - (\vec{B}\cdot\vec{\nabla})B_K$$

$$= -\frac{1}{2}\frac{\partial}{\partial x^K}(B^2) - B_i\frac{\partial B_K}{\partial x^i}$$

Lecture 6 (17.07.12)

Energy relations in general electromagnetic field:

$$\int \vec{E}\cdot\vec{J}\,dv = $$ Work done by the fields on the particles (Conversion of EM energy to mechanical or thermal energy) per unit time

$$= \int \vec{E}\cdot(\vec{\nabla}\times\vec{H})\,dV \pm \int \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\,dv$$

$$= \int\left[\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot\vec{\nabla}\times\vec{E}\right]dv \pm \int \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\,dv$$

$$= -\int \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv \pm \int\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right)dv$$

For vacuum or linear dispersionless dielectric

$$\text{R.H.S} = -\frac{\partial}{\partial t}\left[\frac{1}{2}\int(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\,dv\right] - \int_S(\vec{E}\times\vec{H})\cdot d\vec{s}$$

So
$$\frac{\partial}{\partial t}\left[\frac{1}{2}\int(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\,dv\right] = -\int\vec{E}\cdot\vec{J}\,dv - \int_S(\vec{E}\times\vec{H})\cdot d\vec{s}$$

$\vec{S} = \vec{E}\times\vec{H}$ Poynting vector

Conservation of linear momentum:

The volume force

$$\vec{F_V} = \rho\vec{E} + \vec{J}\times\vec{B}$$

$$= (\epsilon_0 \vec{\nabla}\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$$

$$= \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

$$F_i^V = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E}_i + \frac{1}{\mu_0}[(\vec{\nabla}\times\vec{B})\times\vec{B}]_i - \epsilon_0\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right)_i$$

The first two terms

$$= \epsilon_0(\partial_K E_K)E_i + \frac{1}{\mu_0}\epsilon_{iKl}(\vec{\nabla}\times\vec{B})_K B_l$$

$$= \epsilon_0(\partial_K E_K)E_i + \frac{1}{\mu_0}\epsilon_{iKl}\epsilon_{Kmn}\partial_m B_n B_l$$

$$= \epsilon_0\,\partial_K(E_K E_i) - \epsilon_0 E_K\partial_K E_i + \frac{1}{\mu_0}\epsilon_{iKl}\epsilon_{Kmn}\partial_m B_n B_l$$

$$\cancel{= \epsilon_0\,\partial_K(E_K E_i) - \epsilon_0 E_K\partial_i E_K}$$

$$\epsilon_{lmn}\partial_m E_n = -\frac{\partial B_l}{\partial t}$$

i.e. $\epsilon_{lrs}\epsilon_{lmn}\partial_m E_n = -\epsilon_{lrs}\frac{\partial B_l}{\partial t}$

i.e. $(\delta_{rm}\delta_{sn} - \delta_{rn}\delta_{sm})\partial_m E_n = -\epsilon_{lrs}\frac{\partial B_l}{\partial t}$

or, $\partial_r E_s - \partial_s E_r = -\epsilon_{lrs}\frac{\partial B_l}{\partial t}$

So $\partial_K E_i - \partial_i E_K = -\epsilon_{lKi}\frac{\partial B_l}{\partial t}$

Hence the first two terms

$$= \epsilon_0\partial_K(E_K E_i) - \epsilon_0 E_K\left(\partial_i E_K - \epsilon_{lKi}\frac{\partial B_l}{\partial t}\right)$$

$$+ \frac{1}{\mu_0}(-\delta_{im}\delta_{ln} + \delta_{in}\delta_{lm})\partial_m B_n B_l$$

$$= \epsilon_0 \partial_k (E_i E_k) - \epsilon_0 \partial_i \left(\tfrac{1}{2} E_k E_k\right)$$

$$\pm\, \epsilon_0 \epsilon_{ikl} E_k \frac{\partial B_l}{\partial t} + \frac{1}{\mu_0}\left[(-\partial_i B_l) B_l + \partial_l B_i B_l\right]$$

$$= \frac{\partial}{\partial x_k}\left[\epsilon_0\left(E_i E_k - \frac{1}{2}\delta_{ik} E^2\right) + \frac{1}{\mu_0}\left(B_i B_k - \frac{1}{2}\delta_{ik} B^2\right)\right]$$

$$\pm\, \epsilon_0 \epsilon_{ikl} E_k \frac{\partial B_l}{\partial t}$$

$$= \frac{\partial T_{ik}}{\partial x_k} \pm \epsilon_0 \left[\vec{E} \times \frac{\partial \vec{B}}{\partial t}\right]_i$$

$$\int_V \vec{F}_i^V \, dV = \frac{\partial T_{ik}}{\partial x_k} - \epsilon_0 \frac{\partial}{\partial t} \cancel{\tfrac{\partial}{\partial}} (\vec{E} \times \vec{B})_i$$

$\vec{F}^V$ is the force acting on the sources of EM field per unit time. If $\vec{P}_m$ is the mechanical momentum of all the charged particles

$$\frac{d\vec{P}_m}{dt} + \frac{d}{dt}\int \frac{1}{c^2}(\vec{E} \times \vec{H})\, dv = \int \frac{\partial T_{ik}}{\partial x_k}\, dv = \oint_S T_{ik}\, dS_k$$

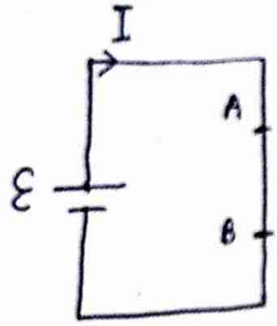
We identify $\frac{1}{c^2}\vec{E} \times \vec{H}$ as the linear momentum per unit time associated with the field. This is necessary to account for the conservation of linear momentum.

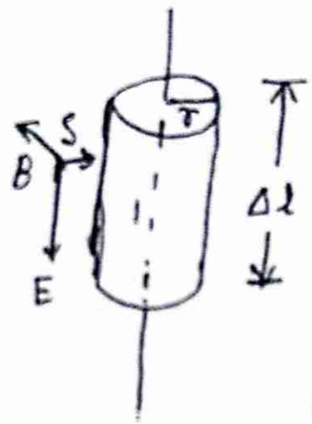
~~Conservation of angular momentum~~

$$\cancel{\vec{F}^{v} = \rho\vec{E} + \vec{J}\times B}$$

$$\cancel{F^{v}_{i} = (\epsilon_0\vec{\nabla}\cdot\vec{E})E_i + [(\vec{\nabla}\times B)\times B]}$$

Energy balance — example





$$E = \frac{\Delta V}{\Delta l} = \frac{IR}{\Delta l}$$

$$B = \frac{\mu_0 I}{2\pi r} \Rightarrow H = \frac{I}{2\pi r}$$

$$S = EH = \frac{I^2R}{\Delta l \cdot 2\pi r}$$

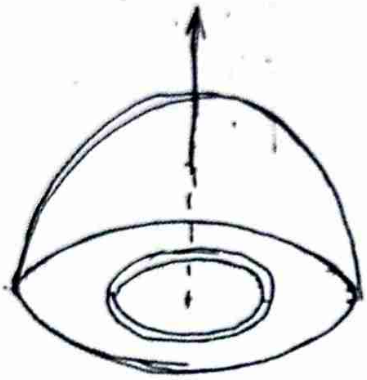
$$\oint \vec{S}\cdot d\vec{s} = \frac{I^2R}{\Delta l \cdot 2\pi r} \cdot 2\pi r\Delta l = I^2R$$

Energy flown in the volume via Poynting vector is equal to the Joule heat loss.

Tutorial – 1 (23.7.12) → Deferred

1. Calculation of force using Maxwell's stress tensor:

Determine the net force on the 'northern hemisphere' of a uniformly charged solid sphere of radius R and charge Q, using Maxwell's stress tensor.



$\underline{r<R}$

$$E(r) = \frac{1}{\epsilon_0}\frac{\frac{4}{3}\pi r^3 \rho}{4\pi r^2} = \frac{4\pi r^3 \rho}{3\epsilon_0 \cdot 4\pi r^2}$$

$$= \frac{\rho r}{3\epsilon_0}$$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$$

$$\therefore \quad E(r) = \frac{r}{3\epsilon_0}\cdot\frac{3Q}{4\pi R^3} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}$$

$\underline{r>R}$ $$E(r) = \frac{1}{\epsilon_0}\frac{Q}{4\pi r^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

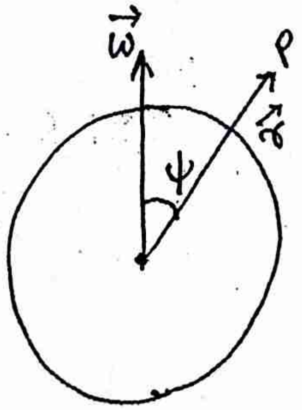
$$F = \int_0^R \frac{\epsilon_0}{2}\frac{1}{16\pi^2\epsilon_0^2}\frac{Q^2 r^2}{R^6}\cdot 2\pi r\,dr + \int_R^\infty \frac{\epsilon_0}{2}\cdot\frac{1}{16\pi^2\epsilon_0^2}\frac{Q^2}{r^4}\cdot 2\pi r\,dr$$

$$= \frac{\epsilon_0 Q^2 \cdot 2\pi}{32\pi^2 \epsilon_0^2 R^6} \cdot \frac{R^4}{4} + \frac{\epsilon_0 Q^2 \cdot 2\pi}{32\pi^2 \epsilon_0^2} \cdot \frac{1}{2R^2}$$

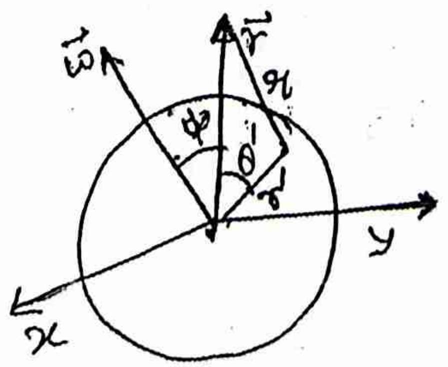
$$= \frac{Q^2}{64\pi\epsilon_0 R^2} + \frac{Q^2}{32\pi\epsilon_0 R^2} = \frac{Q^2}{64\pi\epsilon_0 R^2}(1+2)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{3Q^2}{16 R^2}$$

2. Calculate the force of magnetic attraction between the northern and southern hemispheres of a uniformly charged spinning spherical shell with radius R, angular velocity ω and surface charge density σ.



The vector potential $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(r') ds'}{\mathscr{r}}$



If $r<R$

$$I = \frac{1}{rR}\left[r+R-(R-r)\right] = \frac{2}{R}$$

For $r>R$ $\quad I = \frac{1}{rR}\left[r+R-(r-R)\right] = \frac{2}{r}$

$$\vec{K} = \sigma\vec{V} = \sigma\vec{\omega}\times\vec{r}'$$

$$= \sigma\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= \sigma\Big[-\hat{i}\,\omega\cos\psi\cdot R\sin\theta'\sin\phi' + \hat{j}\,(\omega\cos\psi\cdot R\sin\theta'\cos\phi' - \omega R\sin\psi\cos\theta')$$

$$+\hat{k}\cdot\omega R\sin\psi\sin\theta'\sin\phi'\Big]$$

only the term independent of $\cos\phi'$ or $\sin\phi'$ contribute to $\vec{A}$. So

$$\vec{A} = -\hat{j}\frac{\mu_0}{4\pi}\sigma\omega R\sin\psi\int\frac{2\pi R^2\cos\theta'\sin\theta'\,d\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}$$

$$= -\hat{j}\frac{\mu_0}{2}\sigma R^3\omega\int\frac{\cos\theta'\sin\theta'\,d\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}\sin\psi$$

To evaluate

$$I = \int \frac{\cos\theta' \sin\theta' d\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

Let $r^2 + R^2 - 2rR\cos\theta' = p^2$

$$\Rightarrow \cancel{2} rR \sin\theta' d\theta' = \cancel{2} p\,dp$$

$$I = \frac{1}{2r^2R^2} \int_{\theta=0}^{\pi} \frac{(r^2 + R^2 - p^2)\cancel{p}\,dp}{\cancel{p}}$$

$$= \frac{1}{2r^2R^2} \left[(r^2 + R^2)p - \frac{p^3}{3} \right]_{\theta=0}^{\pi}$$

<u>$r < R$</u> $\quad p = r + R$ at $\theta = \pi$ and $R - r$ at $\theta = 0$

$$I = \frac{1}{2r^2R^2} \left[(r^2 + R^2)(r + R) - \frac{(r+R)^3}{3} \right.$$
$$\left. - (r^2 + R^2)(R - r) + \frac{(R-r)^3}{3} \right]$$

$$= \frac{1}{2r^2R^2} \left[(r^2 + R^2) \cdot 2r + \frac{1}{3}(R^3 - 3R^2r + 3Rr^2 - r^3 - r^3 - R^3 - 3R^2r - 3Rr^2) \right]$$

$$= \frac{1}{2r^2R^2} \left[2r^3 + 2R^2r - \frac{2}{3}(r^3 + 3R^2r) \right]$$

$$= \frac{1}{2r^2R^2} \left[\frac{4r^3}{3} \right] \cancel{- \frac{4R^2r}{3}] - \frac{2}{3rR^2}(r^2 - R^2)}$$

$$= \frac{2r}{3R^2}$$

So $\vec{A} = -\hat{j}\frac{\mu_0}{2}\sigma R^3\omega\sin\psi \times \frac{2r}{3R^2} = -\hat{j}\frac{\mu_0}{3}\sigma R\,\omega r$

i.e. $\vec{A} = \frac{\mu_0}{3}\sigma R(\vec{\omega}\times\vec{r})$

<u>$r > R$</u>

$$I = \frac{1}{2r^2R^2}\Big[(r^2+R^2)\cdot 2R + \frac{1}{3}(r-R-r-R)(r^2+2Rr+R^2-r^2-2Rr-R^2)\Big]$$

$$= \frac{1}{2r^2R^2}\Big[2Rr^2 + 2R^3 + \frac{1}{3}\,2R\cdot 4Rr\Big]$$

$$I = \frac{1}{2r^2R^2}\Big[(r^2+R^2)\cdot 2R + \frac{1}{3}\big(r^3 - 3r^2R + 3rR^2 - R^3 - r^3 - 3r^2R - 3rR^2 - R^3\big)\Big]$$

$$= \frac{1}{2r^2R^2}\Big[2Rr^2 + 2R^3 - \frac{2R^3}{3} - 2r^2R\Big]$$

$$= \frac{1}{2r^2R^2}\cdot\frac{4R^3}{3} = \frac{2R}{3r^2}$$

$$\vec{A} = -\hat{j}\,\frac{\mu_0\sigma R^3\omega}{2}\sin\psi\cdot\frac{2R}{3r^2}$$

$$= \frac{\mu_0\sigma R^4}{3r^3}(\vec{\omega}\times\vec{r})$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

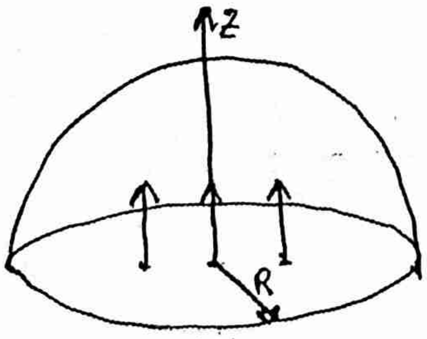
For $r<R$, $\vec{B} = \vec{\nabla} \times \frac{\mu_0 R \sigma}{3}(\vec{\omega} \times \vec{r})$

$$= \frac{\mu_0 R \sigma}{3}\left[-(\vec{\omega} \cdot \vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla} \cdot \vec{r})\right]$$

$$= \frac{\mu_0 R \sigma}{3}\left[-\vec{\omega} + 3\omega\right] = \frac{2\mu_0 R \sigma}{3}\vec{\omega}$$

for $r>R$, $\vec{B} = \vec{\nabla} \times \frac{\mu_0 R^4 \sigma}{3r^3}(\vec{\omega} \times \vec{r})$

$$= \frac{\mu_0 R^4 \sigma}{3r^3} \cdot 2\vec{\omega} = \frac{2\mu_0 R^4 \sigma}{3r^3}\vec{\omega}$$



$$dF_z = T_{zx}\,dS_x + T_{zy}\,dS_y + T_{zz}\,dS_z$$

$$= \frac{1}{\mu_0}\left(B_z^2 - \frac{1}{2}B^2\right)dS$$

$$= \frac{1}{2\mu_0}B^2\,dS$$

For

$$F_z = \frac{1}{2\mu_0}\cdot\frac{4\mu_0^2R^2\sigma^2}{9}\omega^2\cdot\pi R^2 + \frac{1}{2\mu_0}\frac{4\mu_0^2R^8\sigma^2}{9}\omega^2\int_R^\infty \frac{1}{r^8}\cdot 2\pi r\,dr$$

$$-\frac{1}{4}\frac{1}{r^4}$$

$$= \frac{2\mu_0 R^4\pi\sigma^2\omega^2}{9\cancel{\mu_0}} + \frac{4\mu_0 R^8\pi\sigma^2\omega^2}{9}\cdot\left(\cancel{-\frac{1}{2}\cdot\frac{1}{r^4}}\right)\Bigg|_R^\infty$$

$$= \frac{2\mu_0\pi\sigma^2\omega^2}{9}R^4 + \frac{4\mu_0\pi\sigma^2\omega^2}{9}\cdot\frac{R^4}{4}$$

$$= \frac{3\mu_0\pi\sigma^2\omega^2R^4}{9} = \frac{\mu_0}{3}\pi\sigma^2\omega^2R^4$$

$$F = \frac{1}{2\mu_0}\cdot\frac{4\mu_0^2R^2\sigma^2}{9}\omega^2\cdot\pi R^2$$

$$+\frac{1}{2\mu_0}$$

For $r>R$, $\vec{B} = \vec{\nabla}\times\frac{\mu_0R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r})$

$$= \frac{\mu_0R^4\sigma}{3}\left[\frac{1}{r^3}\cdot 2\vec{\omega} - (\vec{\omega}\times\vec{r})\times\frac{-3}{r^4}\cdot\frac{\vec{r}}{r}\right]$$

$$= \frac{\mu_0R^4\sigma}{3}\left[\frac{2}{r^3}\vec{\omega} \pm \frac{3}{r^5}(r^2\omega)\right]$$

$$= -\frac{\mu_0R^4\sigma}{3}\cdot\frac{\vec{\omega}}{r^3} \qquad \cancel{\mu_0R^8}$$

$$\frac{1}{2\mu_0}\cdot\frac{\mu_0^2R^8\sigma^2}{9}\omega^2\times 2\pi\cdot\frac{1}{R^4} = \frac{\mu_0\pi\sigma^2\omega^2R^4}{9}$$

For $r < R$

$$\vec{A} = \frac{\mu_0 \sigma R}{3} (\vec{\omega} \times \vec{r})$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{2\mu_0 \sigma R}{3} \vec{\omega} \quad \left(\text{as } \vec{\nabla} \times (\vec{\omega} \times \vec{r}) = 2\vec{\omega} \right)$$

For $r > R$

$$\vec{B} = \vec{\nabla} \times \left(\frac{\mu_0 \sigma R^4}{3 r^3} (\vec{\omega} \times \vec{r}) \right)$$

$$= \frac{\mu_0 \sigma R^4}{3} \left[\frac{1}{r^3} \vec{\nabla} \times (\vec{\omega} \times \vec{r}) + \vec{\nabla}\left(\frac{1}{r^3}\right) \times (\vec{\omega} \times \vec{r}) \right]$$

$$= \frac{\mu_0 \sigma R^4}{3} \left[\frac{2\vec{\omega}}{r^3} - \frac{3}{r^5} \vec{r} \times (\vec{\omega} \times \vec{r}) \right]$$

$$= \frac{\mu_0 \sigma R^4}{3} \left[\frac{2\vec{\omega}}{r^3} - \frac{3}{r^5} \cdot r^2 \vec{\omega} \right]$$

$$= - \frac{\mu_0 \sigma R^4}{3} \cdot \frac{\vec{\omega}}{r^3}$$

$$F_1 = \frac{1}{2\mu_0} \frac{4\mu_0^2 \sigma^2 R^2}{9} \omega^2 \cdot \pi R^2 = \frac{2\pi \mu_0 \sigma^2 R^4}{9}$$

$$F_2 = \frac{1}{2\mu_0} \cdot \frac{\mu_0^2 \sigma^2 R^8}{9} \omega^2 \cdot \int_R^{\infty} \frac{1}{r^6} \times 2\pi r \, dr$$

$$= \frac{\pi \mu_0 \sigma^2 R^4 \omega^2}{9 \cdot 4}$$

$$F = F_1 + F_2 = \frac{\pi \mu_0 \sigma^2 R^4}{9}\left(2 + \frac{1}{4}\right)\omega^2 = \frac{\pi}{4}\mu_0 \sigma^2 \omega^2 R^4$$

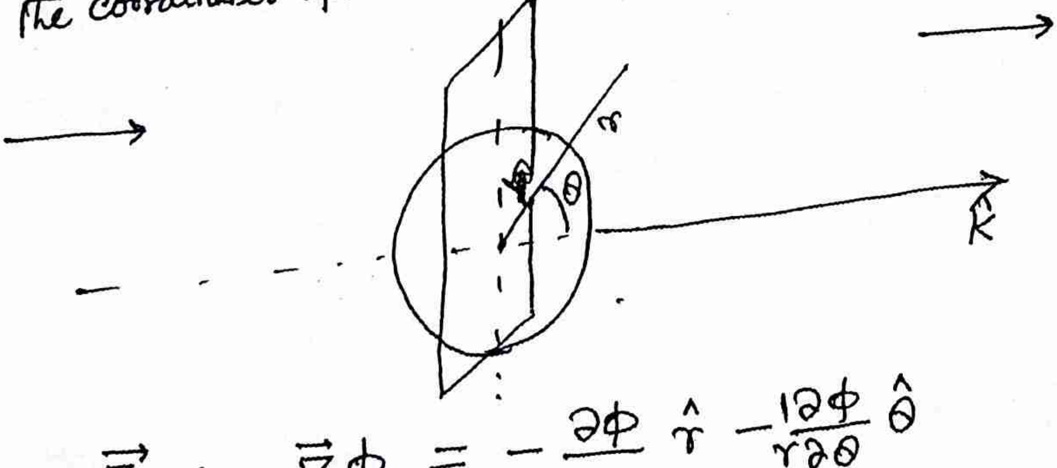
(3) A conducting spherical shell of radius a is placed in a uniform field $\vec{E_0}$. Show that the force tending to separate two halves of the sphere across a diametral plane perpendicular to $\vec{E}$ is given by

$$F = \frac{9}{4}\pi \epsilon_0 a^2 E_0^2$$

If a conducting sphere is placed in a uniform electric field $\vec{E_0}$, the potential is

$$\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

where the origin is at the centre of the sphere and r, θ, ϕ are the coordinates of the observation point.



$$\vec{E_0} = -\vec{\nabla}\phi = -\frac{\partial \phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta}$$

$$= -E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\,\hat{r} + \frac{1}{r}E_0\left(r - \frac{a^3}{r^2}\right)(-\sin\theta)\hat{\theta}$$

$$= -E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\,\hat{r} \pm E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta\,\hat{\theta}$$

$$= -E_0\left(1 + \frac{2a^3}{r^3}\right)$$

On a point P on the diametral plane $\theta = \frac{\pi}{2}$ and

$$\vec{E} = -E_0\left(1 - \frac{a^3}{r^3}\right)\hat{\theta}$$

But here $\hat{\theta} = -\hat{k}$. So $\vec{E} = E_0\left(1 - \frac{a^3}{r^3}\right)\hat{k}$

Inside the spherical shell, $\vec{E} = 0$

$$F = \frac{\epsilon_0}{2}\int_{r=a}^{\infty} E_0^2\left(1 - \frac{a^3}{r^3}\right)^2 2\pi r\,dr$$

$$= \frac{\epsilon_0 E^2}{2}\cdot 2\pi\int\left[1 - \frac{2a^3}{r^3} + \frac{a^6}{r^6}\right] r\,dr$$

Lecture – 7 (23. 7. 12)

The potential formulation of electrodynamics

Consider EM field created in vacuum in presence of sources.

The Maxwell's equations are

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

The scalar potential ϕ and the vector potential $\vec{A}$ can be introduced so as to automatically satisfy the homogenous set of Maxwell equations. From $\vec{\nabla}\cdot\vec{B} = 0$ we can write

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

Since divergence of curl of a vector field vanishes always.

Putting this in $\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t} = 0$, we get

$$\vec{\nabla}\times\vec{E}+\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = 0$$

or, $$\vec{\nabla}\times\vec{E}+\vec{\nabla}\times\frac{\partial\vec{A}}{\partial t} = 0$$

or, $$\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$$

Hence $$\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$ (since curl grad ϕ is always zero)

or, $$\vec{E} = -\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}$$

Thus, if we write

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

The homogenous set of Maxwell equations are always satisfied. To determine ϕ and $\vec{A}$ we take the help of the inhomogenous equation. From $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ we get

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\epsilon_0 \qquad \text{— (1)}$$

From $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}$ we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \epsilon_0\mu_0 \frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)$$

Now $\epsilon_0\mu_0 = \frac{1}{c^2}$. So

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} + \frac{1}{c^2}\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \cancel{0}\,\mu_0\vec{j} \;\cancel{(2)}$$

or, $$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \mu_0\vec{j} \quad \text{— (2)}$$

Both ϕ and $\vec{A}$ are coupled in differential equations (1) & (2). But they may be considerably simplified taking the help of gauge invariance of electrodynamics.

Gauge invariance

If we substitute $\vec{A}$ by

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$

where Λ is any well-behaved function of $\vec{r}$ and t, then $\vec{B}$ remains unchanged. From the expression of $\vec{E}$ we find

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}(\vec{A}' - \vec{\nabla}\Lambda)$$

or, $$\vec{E} = -\vec{\nabla}\left(\phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \vec{A}'}{\partial t}$$

$$= -\vec{\nabla}\phi' - \frac{\partial \vec{A}'}{\partial t}$$

where $\phi' = \phi - \frac{\partial \Lambda}{\partial t}$. Thus if we consider the transformations

$$\left.\begin{aligned} \phi \to \phi' &= \phi - \frac{\partial \Lambda}{\partial t} \\ \vec{A} \to \vec{A}' &= \vec{A} + \vec{\nabla}\Lambda \end{aligned}\right\} \quad \text{———} \quad (3)$$

Then the set of fields $(\vec{E}, \vec{B})$ obtained from $(\phi, \vec{A})$ and $(\phi', \vec{A}')$ are identical. The transformations (3) are called gauge transformations and the invariance of $\vec{E}$ and $\vec{B}$ under them is called gauge invariance. Equations (1) and (2) will be unchanged in form under (3).

orenz gauge

Taking the help of gauge transformations we can decouple (1) and (2). We assume the Lorenz gauge such that

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

Then (1) and (2) become

$$① \rightarrow \quad -\vec{\nabla}^2\phi - \frac{\partial}{\partial t}\left(-\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \rho/\epsilon_0$$

i.e. $$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$② \rightarrow \quad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

Both ϕ and $\vec{A}$ satisfies the inhomogenous wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -g$$

where $g(\vec{r},t)$ is the source term. The Lorenz gauge thus decouples the equations for ϕ and $\vec{A}$ and reduce their equations to the same general and convenient form. But before proceeding, we have to show that the Lorenz gauge is admissible.

Suppose $(\phi, \vec{A})$ be a set of potentials which does not satisfy the Lorenz gauge i.e.

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = \alpha(\vec{r},t) \neq 0.$$

We perform a gauge transformations $(\phi, \vec{A}) \longrightarrow (\phi', \vec{A}')$.

Then

$$\vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial\phi'}{\partial t}$$

$$= \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda + \frac{1}{c^2}\frac{\partial\phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}$$

$$= \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} + \alpha$$

If $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\alpha$, the new set of potentials satisfy the Lorenz gauge. But the equation satisfied by Λ offer solution. So the Lorenz gauge is admissible.

~~Coulomb gauge~~ The condition

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

still does not eliminate the arbitrariness completely. If ~~we~~ we consider gauge transformations for which

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$$

The new set of potentials again satisfy the Lorenz condition.

Coulomb gauge

The Lorenz gauge will be assumed in our subsequent calculation. However, there are other gauge choices in use. A popular choice is the Coulomb gauge

$$\vec{\nabla} \cdot \vec{A} = 0$$

Then $\quad \nabla^2 \phi = -\rho/\epsilon_0$

$$\text{i.e. } \phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho \, dV'}{|\vec{r}-\vec{r}'|}$$

which is the instantaneous coulomb potential. Hence the name is coulomb gauge.

Lecture – 8 (24.7.12)

The inhomogenous wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -g$$

where $\psi = \psi(t, \vec{r})$ is the function to be determined and $g = g(t, \vec{r})$ is a known source function. We expand

$$g(t, \vec{r}) = \int_{-\infty}^{+\infty} g_\omega(\vec{r})\, e^{-i\omega t}\, d\omega$$

and $$\psi(t, \vec{r}) = \int_{-\infty}^{+\infty} \psi_\omega(\vec{r})\, e^{-i\omega t}\, d\omega$$

Substitution gives

$$\int_{-\infty}^{\infty}\left(\nabla^2\psi_\omega(\vec{r}) + \frac{\omega^2}{c^2}\psi_\omega\right)e^{-i\omega t}\,d\omega = -\int_{-\infty}^{+\infty} g_\omega(\vec{r})\,\cancel{e^{-i\omega t}}\,d\omega$$

This leads to

$$\nabla^2\psi_\omega(\vec{r}) + k^2\psi_\omega = -g_\omega(\vec{r}) \quad \text{—— (1)}$$

where k is given by $k = \frac{\omega}{c}$. Equation (1) is now solved by the Green's function technique. If $G(\vec{r}, \vec{r}')$ is the corresponding Green's function

$$\nabla^2 G(\vec{r}, \vec{r}') + k^2 G(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}') \quad \text{— 1(A)}$$

Then $$\psi_\omega(\vec{r}) = \int g_\omega(\vec{r}')\, G(\vec{r}, \vec{r}')\, d^3r'$$

...cept for $\vec{r}=\vec{r}'$, the Green's function satisfies

$$\nabla^2 G(\vec{r},\vec{r}') + k^2 G(\vec{r},\vec{r}') = 0$$

Due to symmetry

$$G(\vec{r},\vec{r}') = G(|\vec{r}-\vec{r}'|) = G(r)$$

where $\vec{r} = \vec{r} - \vec{r}'$. Then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dG}{dr}\right) + k^2 G = 0$$

or, $$\cancel{r}\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} + k^2 G = 0 \qquad \text{———} \qquad (2)$$

Now $$\frac{d^2}{dr^2}(Gr) = \frac{d}{dr}\left(r\frac{dG}{dr} + G\right)$$

$$= r\frac{d^2G}{dr^2} + 2\frac{dG}{dr}$$

So $$\frac{1}{r}\frac{d^2(Gr)}{dr^2} = \frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr}$$

Equation (2) thus becomes

$$\frac{d^2(Gr)}{dr^2} + k^2(Gr) = 0$$

So $$\cancel{G(r)}\; Gr = Ae^{ikr} + Be^{-ikr}$$

Let us write $G^{+}(r) = \dfrac{e^{ikr}}{r}$ and $G^{-}(r) = \dfrac{e^{-ikr}}{r}$ i.e.

$$G(r) = A\,G^{+}(r) + B\,G^{-}(r)$$

As $r \to 0$, both $G^{+}(r)$ and $G^{-}(r) \to \dfrac{1}{r}$ i.e.

$$G(r) \xrightarrow{r \to 0} \frac{A+B}{r}$$

But as $\vec{r} \to \vec{r}\,'$ equation 1(A) reduces to Poisson's equation

$$\nabla^2 G(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,')$$

So $G(\vec{r},\vec{r}\,') \cancel{\dfrac{1}{4\pi}} = \dfrac{1}{4\pi|\vec{r}-\vec{r}\,'|} = \dfrac{1}{4\pi R}$

So $A + B = \dfrac{1}{4\pi}$

Hence

$$\psi_\omega(\vec{r}) = \int g_\omega(\vec{r}\,')\cdot\left(A\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} + B\,\frac{e^{-ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\right) d^3r'$$

Substituting

$$\psi(t,\vec{r}) = \iint g_\omega(\vec{r}\,')\left[A\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} + B\,\frac{e^{-ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\right] e^{-i\omega t}\, d^3r'\, d\omega$$

i.e. $$\psi(t,\vec{r}) = A\iint \frac{g_\omega(\vec{r}')}{|\vec{r}-\vec{r}'|}\, e^{-i\omega\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)} d\omega\, d^3r'$$

$$+ B\iint \frac{g_\omega(\vec{r}')}{|\vec{r}-\vec{r}'|}\, e^{-i\omega\left(t+\frac{|\vec{r}-\vec{r}'|}{c}\right)} d\omega\, d^3\vec{r}'$$

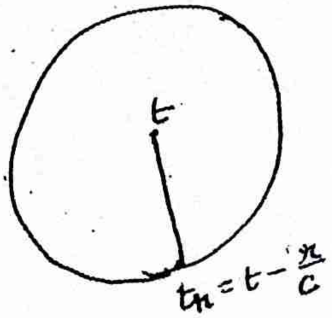
$$= A\int \frac{g(t-\frac{r}{c},\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r' + B\int \frac{g(t+\frac{r}{c},\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'$$

The first solution shows that the source at $\vec{r}'$ influences at $\vec{r}$ is that at an earlier time $t-\frac{r}{c}$, precisely the time ~~diff~~ ~~amount~~ earlier which takes electromagnetic signal to reach $\vec{r}$. This is the causal solution. The second part is the advanced solution which is acausal. We will consider only the first part i.e. the retarded solution. So $B=0$ and $A=\frac{1}{4\pi}$. The retarded ~~solution~~ for $\psi(t,\vec{r})$ is then

$$\psi(t,\vec{r}) = \frac{1}{4\pi}\int \frac{g(t_r,\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'$$

where $$g(t_r,\vec{r}') = \int g_\omega(\vec{r}')\, e^{-i\omega(t-\frac{r}{c})} d^3r'$$

The retarded solution can be physically understood by the concept of the information collecting sphere



From the solution to the inhomogenous wave equation we can write

$$\phi(t,\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{[\rho(\vec{r}\,')]}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

$$\vec{A}(t,\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{[\vec{J}(\vec{r}\,')]}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

It is interesting to verify that the above expressions for ϕ and $\vec{A}$ satisfies the Lorentz condition

$$\vec{A} = \frac{\mu_0}{4\pi}\int\int \frac{\vec{j}_\omega(\vec{r}')\, e^{-i\omega(t-\frac{r}{c})}}{|\vec{r}-\vec{r}'|}\, d\omega\, d^3r'$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\mu_0}{4\pi}\int\int \vec{j}_\omega(\vec{r}')\cdot\vec{\nabla}\left(\frac{e^{-i\omega(t-\frac{r}{c})}}{|\vec{r}-\vec{r}'|}\right) d\omega\, d^3r'$$

$$= -\frac{\mu_0}{4\pi}\int\int \vec{j}_\omega(\vec{r}')\cdot\vec{\nabla}'\left(\frac{e^{-i\omega(t-r/c)}}{|\vec{r}-\vec{r}'|}\right) d\omega\, d^3r'$$

$$= \frac{\mu_0}{4\pi}\int\int \vec{\nabla}\cdot\vec{j}_\omega(\vec{r}')\,\frac{e^{-i\omega(t-r/c)}}{|\vec{r}-\vec{r}'|}\, d\omega\, d^3r'$$

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} = \frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int\int \phi_\omega(\vec{r}')\times(-i\omega)\times\frac{e^{-i\omega(t-\frac{r}{c})}}{|\vec{r}-\vec{r}'|}\, d\omega\, d^3r'$$

$$= -\frac{i\mu_0}{4\pi}\int\int \phi_\omega(\vec{r}')\,\omega\,\frac{e^{-i\omega(t-r/c)}}{|\vec{r}-\vec{r}'|}\, d\omega\, d^3r'$$

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}$$

$$= -\frac{\mu_0}{4\pi}\int\int(\vec{\nabla}\cdot\vec{j}_\omega - i\omega\phi_\omega)\,\frac{e^{-i\omega(t-r/c)}}{|\vec{r}-\vec{r}'|}\, d\omega\, d^3r'$$

From the equation of continuity

$$\int(\vec{\nabla}\cdot\vec{j}_\omega - i\omega\phi_\omega)\, e^{-i\omega t}\, d\omega = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{j}_\omega - i\omega\phi_\omega = 0$$

Hence

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

The retarded solutions satisfy the Lorenz gauge.