Unit - 3

Infinite Series of real numbers:-

Let $\{x_n\}$ be a seqn. of real nos. Then the expression $x_1+x_2+\cdots+x_n+\cdots$ or $\sum_{n=1}^{\infty} x_n$ or $\sum x_n$ is called an infinite series of real nos.

$x_1, x_2, \ldots, x_n, \ldots$ are called the terms of the series; for every $n \in N$, x_n is called the n^{th} term of the series; the seqn. $\{S_n\}$, where $S_n = x_1+x_2+\cdots+x_n$, $\forall n \in N$, is called the seqn. of partial sums of the series.

- An infinite series $\sum_{n=1}^{\infty} x_n$ of real nos. is said to be (i) convergent if the seqn. $\{S_n\}$ of partial sums of the series is convergent. If $\lim_{n\to\infty} S_n = S$, we say that S is the limit of a sum of $\sum x_n$, or S is the sum of $\sum x_n$ and we write $\sum x_n = S$. In this case, we also say that the series $\sum x_n$ converges to S.
  (ii) diverges to $\pm\infty$ if $\lim_{n\to\infty} S_n = \pm\infty$.
  (iii) Oscillates finitely or infinitely if $\{S_n\}$ oscillates finitely or infinitely respectively.

- Ex:- Test the convergence of the following series
(i) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (ii) $\sum_{n=1}^{\infty} n$ (iii) $\sum_{n=1}^{\infty} (-1)^{n+1}$ (iv) $\sum_{n=1}^{\infty} (-1)^{n+1} n$.

Soln (i) For all $n \in N$, we have

$$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)}$$
$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1}$$
$$= 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty.$$

$\therefore$ The given series is convergent and its sum is 1.

(ii) $S_n = 1+2+\cdots+n = \frac{n(n+1)}{2} \to \infty$ as $n \to \infty$

$\therefore$ The given series is divergent and diverges to ∞

(iii) $S_n = 1-1+1-1+\cdots$ upto n^{th} term.

$$= \begin{cases} 1, & \text{if } n \text{ be odd} \\ 0, & \text{if } n \text{ be even} \end{cases}$$

$\therefore$ $\{S_n\}$ oscillates finitely between 0 and 1

$\therefore$ The given series oscillates finitely between 0 and 1.

(iv) $S_n = 1-2+3-4+5-6+\cdots$ upto n^{th} term

$$= \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ -\frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$, which oscillates infinitely

$\therefore$ The given series oscillates infinitely.

• **Ex:-** Test the convergence of the Geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ $(a>0)$.

Soln:- We shall show that the series is
(i) convergent if $|r|<1$ (ii) divergent if $r \geq 1$ (iii) oscillates finitely if $r=-1$ and (iv) oscillates infinitely if $r<-1$.

For every $n \in \mathbb{N}$, we have
$S_n = a + ar + \cdots + ar^{n-1} = a \cdot \frac{r^n - 1}{r-1}$, provided $r \neq 1$.

(i) If $|r|<1$, then $r^n \to 0$ as $n \to \infty$ so that $S_n \to \frac{a}{1-r}$ as $n \to \infty$
$\therefore$ In this case the series is convergent and its sum is $\frac{a}{1-r}$.

(ii) If $r=1$, then for every $n \in \mathbb{N}$, $S_n = a + a + \cdots + a$ (n-times)
$= na \to \infty$ as $n \to \infty$ ($\because a>0$)
$\therefore$ The series is divergent and diverges to ∞ for $r=1$.
If $r>1$, then $r^n \to \infty$ as $n \to \infty$ so that $S_n \to \infty$ as $n \to \infty$
$\therefore$ The series is divergent and diverges to ∞ for $r>1$.
$\therefore$ The series is divergent for $r \geq 1$.

(iii) If $r=-1$, then for every $n \in \mathbb{N}$,
$S_n = a - a + a - a + \cdots$ (n-times)
$= \begin{cases} a, & \text{if } n \text{ be odd} \\ 0, & \text{if } n \text{ be even} \end{cases}$, which oscillates finitely between 0 and a
$\therefore$ ~~In this case~~ The series oscillates finitely between 0 and a for $r=-1$.

(iv) If $r<-1$, then r^n oscillates infinitely ~~between~~ so that $\{S_n\}$ oscillates infinitely. $\therefore$ The series oscillates infinitely for $r<-1$

• **Ex:-** Test the convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

We shall show that the series is (i) convergent if $p>1$ and (ii) divergent if $p \leq 1$.

(i) Let $p>1$. $\because$ All the terms of the series is positive, hence the seqn. $\{S_n\}$ of partial sums of the series is m.i.
($\because \{S_n\}$ is m.i.)
Now, $\forall n \in \mathbb{N}$, we have

$$0 < S_n \leq S_{2^n-1} = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p}\right) + \cdots$$
$$\cdots + \left\{\frac{1}{(2^{n-1})^p} + \frac{1}{(2^{n-1}+1)^p} + \cdots + \frac{1}{(2^n-1)^p}\right\}$$
$$\leq 1 + 2 \times \frac{1}{2^p} + 4 \times \frac{1}{4^p} + 8 \times \frac{1}{8^p} + \cdots + 2^{n-1} \times \frac{1}{(2^{n-1})^p}$$
$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^2)^{p-1}} + \frac{1}{(2^3)^{p-1}} + \cdots + \frac{1}{(2^{n-1})^{p-1}}$$
$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \frac{1}{(2^{p-1})^3} + \cdots + \frac{1}{(2^{p-1})^{n-1}}$$
$$= \frac{1 - \left(\frac{1}{2^{p-1}}\right)^n}{1 - \frac{1}{2^{p-1}}} < \frac{1}{1 - \frac{1}{2^{p-1}}}$$

$\therefore \{S_n\}$ is bounded.

$\therefore \{S_n\}$ is m.i. and bounded. Hence $\{S_n\}$ is convergent

$\therefore$ The series is convergent, ~~in this case~~ if $p>1$.

(ii) Let $p \le 1$. Then $n^p \le n$, $\forall n \in \mathbb{N}$ so that $\frac{1}{n^p} \ge \frac{1}{n}$, $\forall n \in \mathbb{N}$ — ①

Let $G>0$ be given arbitrarily.

Now, $S_{2^n} = 1 + \frac{1}{2^p} + \left(\frac{1}{3^p}+\frac{1}{4^p}\right) + \left(\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p}\right) + \left(\frac{1}{9^p}+\frac{1}{10^p}+\cdots+\frac{1}{16^p}\right)$

$+ \cdots + \left(\frac{1}{(2^{n-1}+1)^p} + \frac{1}{(2^{n-1}+2)^p} + \cdots + \frac{1}{(2^n)^p}\right)$

$\ge 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)$

$+ \cdots + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right)$

$\ge 1 + \frac{1}{2} + 2\times\frac{1}{4} + 4\times\frac{1}{8} + 8\times\frac{1}{16} + \cdots + 2^{n-1}\times\frac{1}{2^n}$

$= 1 + \frac{n}{2} > G$, whenever $n > 2(G-1)$.

Again, $\{S_n\}$ is m.i.

$\therefore S_n > G$, $\forall n \ge n_0$, where n_0 is the smallest positive integer $> 2^{2(G-1)}$.

$\therefore \{S_n\}$ is m.i. and unbounded above. Hence $\{S_n\}$ is divergent and diverges to ∞.

• Th [Cauchy's convergence criterion for infinite series]

An infinite series $\sum_{n=1}^{\infty} x_n$ converges iff for any given $\varepsilon>0$, $\exists n_0 \in \mathbb{N}$ s.t. $|x_{n+1}+x_{n+2}+\cdots+x_{n+p}| < \varepsilon$, $\forall n \ge n_0$, $\forall p \ge 1$, i.e.,

$|S_{n+p} - S_n| < \varepsilon$, $\forall n \ge n_0, \forall p \ge 1$, where $S_n = x_1+x_2+\cdots+x_n$, $\forall n \in \mathbb{N}$.

Pf:- ✓

• Th:- If $\sum_{n=1}^{\infty} x_n$ converges then $\lim_{n\to\infty} x_n = 0$ but not conversely.

Pf:- Let $\sum_{n=1}^{\infty} x_n$ be convergent. Then by Cauchy's criterion, we have, for any given $\varepsilon>0$, $\exists n_0 \in \mathbb{N}$ s.t.

$|x_{n+1}+x_{n+2}+\cdots+x_{n+p}| < \varepsilon$, $\forall n \ge n_0$, $\forall p \ge 1$.

$\therefore$ Taking $p=1$, we have $|x_{n+1}| < \varepsilon$, $\forall n \ge n_0$. This proves that $\lim_{n\to\infty} x_n = 0$.

To disprove the converse part, consider the series $\sum_{n=1}^{\infty} x_n$, where $x_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

But the series $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, as it is a p-series with $p=1$.

• **Note:-** From the last theorem, it is clear that, for an infinite series $\sum_{n=1}^{\infty} x_n$, if $\lim_{n\to\infty} x_n \neq 0$ or if $\{x_n\}$ is not convergent, then $\sum_{n=1}^{\infty} x_n$ is not convergent.

• **Probl:-** Show that (i) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ does not converge
(ii) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ does not converge.

Soln:- (i) $x_n = \frac{n^n}{n!} = \frac{n.n.n.\cdots n}{1.2.3.\cdots n} > 1, \forall n > 1.$

$\therefore \lim_{n\to\infty} x_n \geq 1 > 0$ or $\{x_n\}$ does not converge.

$\therefore \sum x_n$ does not converge.

(ii) $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1 \neq 0.$ $\therefore \sum \frac{n}{n+1}$ does not converge.

• **Probl:-** Using Cauchy criterion, prove that
(i) $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges (ii) $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

Let $\varepsilon > 0$ be given arbitrarily.

(i) Now, $\left|\sum_{k=n+1}^{n+p} \frac{1}{k!}\right| = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+p)!}$

$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p-1}}$

$= \frac{1}{2^n} \cdot \frac{1-(\frac{1}{2})^p}{1-\frac{1}{2}} < \frac{1}{2^n} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{n-1}}, \forall p \geq 1.$

$\therefore \left|\sum_{k=n+1}^{n+p} \frac{1}{k!}\right| < \frac{1}{2^{n-1}} < \varepsilon, \forall n \geq n_0, \forall p \geq 1$, where n_0 is the smallest +ve integer $> \frac{\log(\frac{1}{\varepsilon})}{\log 2} + 1$

$\boxed{2^{n-1} > \frac{1}{\varepsilon} \quad n-1 >}$

$\therefore$ By Cauchy criterion, the given series is convergent.

(ii) Consider $\varepsilon = \frac{1}{4} > 0$.

Now, $\left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}\right| = \left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right|$ (taking $p = n$)

$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq n \times \frac{1}{2n} = \frac{1}{2} > \frac{1}{4} = \varepsilon, \forall n$ and $p = n$.

$\therefore$ By Cauchy criterion, the given series does not converge.

• **Note:-** If a series $\sum x_n$ be convergent or divergent and we add or remove finitely many terms to or from this series, then the new series is also convergent or divergent respectively. Particularly, if $\sum_{n=1}^{\infty} x_n$ converges to S and n_0 be a fixed positive integer, then $\sum_{n=n_0+1}^{\infty} x_n$ converges to $S-(x_1+x_2+\cdots+x_{n_0})$

• **Th (Abel's Theorem):** If $\{x_n\}$ is monotonic sequence of +ve nos. and $\sum_{n=1}^{\infty} x_n$ is convergent then $\lim_{n\to\infty} n x_n = 0$, but not conversely.

Pf:- Let $\varepsilon > 0$ be given arbitrarily. Since $\sum_{n=1}^{\infty} x_n$ is convergent,
$\exists\, n_0 \in \mathbb{N}$ s.t. $|x_{n_0+1} + x_{n_0+2} + \cdots + x_n| < \varepsilon/2, \forall n > n_0$.

i.e., $x_{n_0+1}+x_{n_0+2}+\cdots+x_n<\varepsilon/2,\ \forall n>n_0$ —— (1) ($\because x_n>0,\ \forall n\in\mathbb{N}$).

Since $\{x_n\}$ is m.d., hence $x_{n+1}\le x_n,\ \forall n\in\mathbb{N}$ —— (2)

From (1) & (2), we get

$(n-n_0)x_n \le x_{n_0+1}+x_{n_0+2}+\cdots+x_n<\varepsilon/2,\ \forall n>n_0$.

$\Rightarrow nx_n<\varepsilon/2+n_0x_n,\ \forall n>n_0$ —— (3).

$\because \sum x_n$ is convergent, hence $\lim\limits_{n\to\infty}x_n=0$. Therefore $\exists\, n_1\in\mathbb{N}$

s.t. $|x_n|=x_n<\varepsilon/2n_0,\ \forall n>n_1$ —— (4)

From (3) & (4), we get

$nx_n<\varepsilon/2+\varepsilon/2=\varepsilon,\ \forall n>n_2=\max\{n_0,n_1\}$.

$\therefore \lim\limits_{n\to\infty} nx_n=0$.

To disprove the converse part, we consider the series $\sum\limits_{n=2}^{\infty}\frac{1}{n\log n}$ which is divergent, but $\lim\limits_{n\to\infty} n\times\frac{1}{n\log n}=0$.

• **Note:-** From the last theorem it is clear that, for a m.d. seqn. $\{x_n\}$ of positive nos., if $\lim\limits_{n\to\infty} nx_n\ne 0$, then $\sum x_n$ does not converge.

• **Th:-** If m be a fixed +ve integer, then ~~both~~ the series $\sum\limits_{n=1}^{\infty}x_n$ and $\sum\limits_{n=m+1}^{\infty}x_n$ are both convergent or both divergent or both oscillatory series. Also, if $\sum\limits_{n=1}^{\infty}x_n$ converges to S, then $\sum\limits_{n=m+1}^{\infty}x_n$ converges to $S-(a_1+a_2+\cdots+a_m)$.

Pf:- Proof is not required.

Th:- If $\sum x_n$ & $\sum y_n$ converge to x & y, then $\sum(x_n+y_n)$ converges to $x+y$.

• **Th:-** If λ be any non-zero constant, then $\sum x_n$ and $\sum(\lambda x_n)$ are both convergent or both divergent or both oscillatory seqn. Also, if $\sum x_n$ converges to s, then $\sum(\lambda x_n)$ converges to λs.

Pf:- Proof is not required.

Th:- If the terms of a convergent series are grouped in parenthesis in any manner to form new terms (the order of the terms remains unaltered) then the resulting series will converge to the same sum.

Tests of convergence and divergence of infinite series of non-negative terms:

• Note:- (At last) If $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=0$ and $\sum y_n$ converges then $\sum x_n$ converges.
If $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=\infty$ and $\sum y_n$ diverges then $\sum x_n$ diverges.

1. Comparison Tests:-

(A) [In terms of upper and lower limit]
Let $\sum x_n$ and $\sum y_n$ be two series of non-negative terms.

(i) If $\sum y_n$ is known to be convergent and $\overline{\lim}\frac{x_n}{y_n}<\infty$, then $\sum x_n$ is convergent.

(ii) If $\sum y_n$ is known to be divergent and $\underline{\lim}\frac{x_n}{y_n}>0$, then $\sum x_n$ is divergent.

• **Note:-** In most of the cases, $\lim\limits_{n\to\infty}\frac{x_n}{y_n}$ exists and equal to l (say), so that $\overline{\lim}\frac{x_n}{y_n}=\underline{\lim}\frac{x_n}{y_n}=l$. Let $0<l<\infty$. Then either both $\sum x_n$ and $\sum y_n$ converge or both diverge. [This is called "Limit form of Comparison Test"]

(B) Let Σx_n and Σy_n be two series of non-negative terms

(i) ~~If~~ If Σy_n is known to be convergent and if $\exists$ a positive constant k and $n_0 \in \mathbb{N}$ s.t. $x_n \leq k y_n$, $\forall n \geq n_0$, then Σx_n is convergent.

(ii) If Σy_n is known to be divergent and if $\exists$ a positive constant k and $n_0 \in \mathbb{N}$ s.t. $x_n \geq k y_n$, $\forall n \geq n_0$, then Σx_n is divergent.

(C) Let Σx_n and Σy_n be series of non-negative terms and $\exists$ $n_0 \in \mathbb{N}$ s.t. $\frac{x_n}{x_{n+1}} \geq \frac{y_n}{y_{n+1}}$, $\forall n \geq n_0$. Then

(i) Σx_n is convergent if Σy_n is convergent.

(ii) Σy_n is divergent if Σx_n is divergent.

• Probl! – ~~The~~ Test the convergence of the series

(i) $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \frac{4}{7.9} + \cdots \infty$ (divergent)

(ii) $\frac{5}{1.2.4} + \frac{7}{2.3.5} + \frac{9}{3.4.6} + \frac{11}{4.5.7} + \cdots \infty$ (Conv.)

(iii) $\sum_{n=2}^{\infty} \frac{\log n}{\sqrt{n+1}}$ (divergent) (Compare with $\Sigma \frac{1}{\sqrt{n}}$)

(iv) $\sum_{n=1}^{\infty} \left((n^3+1)^{1/3} - n\right)$ (Convergent)

(Compare by $\Sigma \frac{1}{n}$).

(v) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ (,, ,, $\Sigma \frac{1}{n}$)

(vi) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(vii) $\frac{1.2}{3^2.4^2} + \frac{3.4}{5^2.6^2} + \frac{5.6}{7^2.8^2} + \cdots$ (Conv.)

(viii) $\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots + \frac{1+2+\cdots+n}{n^3} + \cdots \infty$ (div)

Soln! – (i) $x_n = \frac{n}{(2n-1)(2n+1)}$, $\forall n \in \mathbb{N}$

$> \frac{n}{2n.3n} = \frac{1}{6n} = y_n$ (say) $\forall n \in \mathbb{N}$.

(ii) $x_n = \frac{2n+3}{n(n+1)(n+3)}$, $\forall n \in \mathbb{N}$.

Consider $y_n = \frac{1}{n^2}$, $\forall n \in \mathbb{N}$

Now, $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^2(2n+3)}{n(n+1)(n+3)} = 2$, a positive finite no.

$\therefore$ Σx_n and Σy_n are both conv. or both div.

But $\Sigma y_n = \Sigma \frac{1}{n^2}$ is conv. $\therefore$ Σx_n is conv.

(iii) $x_n = \frac{\log n}{\sqrt{n+1}}$, $n > 1$. Consider $y_n = \frac{1}{\sqrt{n}}$.

$\lim_{n\to\infty} \frac{x_n}{y_n} = \infty > 0$ Now Σy_n is div. $\therefore$ Σx_n is div.

(iv) $x_n = (n^3+1)^{1/3} - n = \dfrac{n^3+1-n^3}{(n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2}$

Let $y_n = \frac{1}{n^2}$.

Now, $\lim_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{n^2}{(n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^2} = \dfrac{1}{(1+\frac{1}{n^3})^{2/3}+(1+\frac{1}{n^3})^{1/3}+1}$

$\to \frac{1}{3}$ as $n\to\infty$, which is positive and finite.

Now $\sum y_n$ is conv. $\therefore \sum x_n$ is conv.

(v) $x_n = \dfrac{1}{n^{1+\frac{1}{n}}}$. Let $y_n = \frac{1}{n}$.

$\therefore \lim_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{1}{n^{1/n}} \to 1$ as $n\to\infty$, which is +ve. ~~and finite~~

Now $\sum y_n$ is div. $\therefore \sum x_n$ is div.

(vi) $x_n = \sin\frac{1}{n}$. Let $y_n = \frac{1}{n}$.

Now, $\lim_{n\to\infty} \dfrac{x_n}{y_n} = \lim_{n\to\infty} \dfrac{\sin\frac{1}{n}}{1/n} = 1$ which is ~~finite and~~ +ve.

Now $\sum y_n$ is div. $\therefore \sum x_n$ is div.

(vii) $x_n = \dfrac{(2n-1).2n}{(2n+1)^2.(2n+2)^2}$. Let $y_n = \frac{1}{n^2}$.

Now, $\lim_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{(2n-1).2n.n^2}{(2n+1)^2(2n+2)^2} \to \dfrac{2\times 2}{4\times 4} = \frac{1}{4}$ which is +ve and finite.

Now $\sum y_n$ is conv. $\therefore \sum x_n$ is conv.

(viii) $x_n = \dfrac{1+2+\cdots+(n+1)}{(n+1)^2} = \dfrac{(n+1)(n+2)}{(n+1)^2} = \dfrac{n+2}{n+1} > 1 = y_n$.

Now, $\sum y_n$ is div. $\therefore \sum x_n$ is div.

~~Now $\lim_{n\to\infty}\frac{x_n}{y_n}$~~

• Prob:- If $\sum x_n$ converges and $x_n > 0, \forall n\in\mathbb{N}$, prove that $\sum x_n^p$ also converges for $p>1$.

Soln:- $\sum x_n$ is conv. $\Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow$ for $\varepsilon = \frac{1}{2}$, $\exists\, n_0\in\mathbb{N}$ s.t.

$|x_n| < \frac{1}{2}, \forall n \geq n_0 \Rightarrow$ ~~that is a proper fraction~~ $|x_n^p| < |x_n|, \forall n\geq n_0$, for $p>1$.

$\Rightarrow x_n^p < x_n, \forall n\geq n_0$ ($\because x_n>0, \forall n\in\mathbb{N}$), for $p>1$.

$\Rightarrow \sum x_n^p$ is conv., by Comparison Test.

• **Cauchy's Root Test:-**

Let $\sum x_n$ be a series of non-negative terms and let $\rho = \overline{\lim}(x_n)^{1/n}$ (or $\rho = \lim_{n\to\infty}(x_n)^{1/n}$, when $\{(x_n)^{1/n}\}$ is convergent).

Then (i) if $\rho<1$, then $\sum x_n$ converges

(ii) if $\rho>1$, then $\sum x_n$ diverges.

If $\rho = 1$, then no definite conclusion can be drawn.

(For examples, $\rho = 1$ for both $\sum\frac{1}{n}$ and $\sum\frac{1}{n^2}$ and $\sum\frac{1}{n}$ is divergent, whereas $\sum\frac{1}{n^2}$ is convergent)

• ~~Raab~~ Raabe's Test:- If $\sum x_n$ is a series of positive terms s.t. $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = l$, then (i) $\sum x_n$ is convergent if $l>1$, and (ii) $\sum x_n$ is divergent if $l<1$.

If $l=1$ then no definite conclusion can be drawn.

1 • **Probl:-** Test the Convergence of the series

(i) $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots+\left(\frac{n}{2n+1}\right)^n+\cdots\infty$ (Conv.)

(ii) $\sum_{n=1}^{\infty} 2^{-n-(-1)^n}$ (Conv.)

(iii) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots\infty$ (Conv)

(iv) $\sum\left(1+\frac{1}{n}\right)^{-n^2}$ (v) $\sum \frac{x^n}{n^n}$ $(x>0)$ (vi) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$ (Conv.)

(vii) $\frac{1}{2}+\frac{2x}{3}+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\cdots$ $(x>0)$

(viii) $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$. (ix) $\sum\frac{(1+nx)^n}{n^n}$ $(x>0)$

(x) $\frac{1^3}{3}+\frac{2^3}{3^2}+1+\frac{4^3}{3^4}+\cdots$ (xi) $\sum\frac{(n+1)^n x^n}{n^{n+1}}$. $(x>0)$

Soln:- (i) $x_n=\left(\frac{n}{2n+1}\right)^n$ $\therefore x_n>0, \forall n\in\mathbb{N}$.

$\therefore x_n^{1/n}=\frac{n}{2n+1}\to\frac{1}{2}$ as $n\to\infty$ and $\frac{1}{2}<1$

$\therefore$ Convergent.

(ii) $x_n^{1/n}=2^{-1-\frac{(-1)^n}{n}}\to 2^{-1}<1$, as $n\to\infty$. $\therefore$ Convergent

(iii) $x_n^{1/n}=\left(\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right)^{-1}=\left(\frac{n+1}{n}\right)^{-1}\left(\left(1+\frac{1}{n}\right)^n-1\right)^{-1}\to\frac{1}{e-1}<1$, as $n\to\infty$

$\therefore$ Convergent

(iv) $x_n^{1/n}=\left(1+\frac{1}{n}\right)^{-n}\to\frac{1}{e}<1$ as $n\to\infty$ $\therefore$ Convergent.

(v) $x_n^{1/n}=\frac{x}{n}\to 0$ as $n\to\infty$ and $0<1$. $\therefore$ Convergent

(vii) $x_n=\left(\frac{n}{n+1}\right)^{n-1}x^{n-1}$

$\therefore x_n^{1/n}=\left(\frac{n}{n+1}\right)^{-\frac{1}{n}}\frac{1}{\left(1+\frac{1}{n}\right)}x^{1-1/n}\to x$ as $n\to\infty$.

$\therefore$ Conv. if $x<1$, div. if $x>1$

If $x=1$, then $x_n=\left(\frac{n}{n+1}\right)^{-1}\times\frac{1}{\left(1+\frac{1}{n}\right)^n}\to\frac{1}{e}$ as $n\to\infty$. and $\frac{1}{e}\neq 0$.

$\therefore$ For $x=1$, $\sum x_n$ divergent

(viii) $x_n^{1/n}=\left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}\to\frac{1}{e}(<1)$ as $n\to\infty$) $\therefore$ Convergent

(ix) $x_n^{1/n}=\frac{1+nx}{n}\to x$ as $n\to\infty$

$\therefore$ Conv. if $x<1$, div. if $x>1$.

If $x=1$, then $x_n=\left(\frac{1+n}{n}\right)^n=\left(1+\frac{1}{n}\right)^n\to e\ (\neq 0)$, as $n\to\infty$

$\therefore$ For $x=1$, $\sum x_n$ is div.

(x) $x_n=\frac{n^3}{3^n}$ $\therefore x_n^{1/n}=\frac{n^{3/n}}{3}\to\frac{1}{3}(<1)$ as $n\to\infty$

$\therefore$ Conv.

(xi) $x_n^{1/n}=\frac{n+1}{n}\times\frac{1}{n^{1/n}}\cdot x\to x$ as $n\to\infty$. $\therefore$ Conv. if $0<x<1$, div. if $x>1$

If $x=1$, then $x_n=\frac{(n+1)^n}{n^{n+1}}$. Let $y_n=\frac{1}{n}$.

$\therefore \lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{(n+1)^n}{n^n}=e$ which is non-zero and finite.

$\therefore \sum x_n$ is div., since $\sum y_n$ is div.

For Hints see bottom of page

• Some more problems on Comparison Test

• Prob: Test the convergence of the series

(i) $\frac{1}{3^2} \cdot \frac{2}{4^2} + \frac{3}{5^2} \cdot \frac{4}{6^2} + \frac{5}{7^2} \cdot \frac{6}{8^2} + \cdots$

(ii) $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$

(iii) $\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \cdots$

(iv) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \cdots$

(v) $\sum_{n=1}^{\infty} \frac{n+3}{n^3-n+1}$ (vi) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+1}$

(vii) $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots$

(viii) $\sum_{n=1}^{\infty} (\sqrt{n^3+1} - \sqrt{n^3})$ (ix) $\sum ((n^3+1)^{1/3} - n)$ (x) $\sum \frac{1}{\sqrt{n}} \sin\frac{1}{n}$

(xi) $\sum \sin\left(\frac{1}{n^p}\right)$ (xii) $\sum \tan^{-1}\left(\frac{1}{n}\right)$ (xiii) $\sum_{n=2}^{\infty} \frac{1}{\log n}$ $(v_n = \frac{1}{n})$

• (xiv) $\sum_{n=2}^{\infty} (\log(n+1) - \log n)$ (Take $v_n = \frac{1}{n}$), (xv) $\sum_{n=2}^{\infty} \frac{\log n}{2n^3-1}$ $(v_n = \frac{1}{n^2})$, (xvi) $\sum e^{-n^2}$ (xvii) $\sum \sin\left(\frac{1}{n}\right)$

(xviii) $\sum \frac{1}{n(n+1)^2}$ (xix) $\sum \frac{1}{n^p (n+1)^p}$ (xx) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \cdots$ (D). (xxi) $\sum \frac{(\log n)^2}{n^p}$

(xxii) $\sum \frac{1}{n^p} \sin\left(\frac{1}{n}\right)$

(xxiii) $\sum \cos\frac{1}{n}$

• Ass: If $\sum x_n$ converges, prove that $\sum \frac{x_n}{1-x_n}$ also conv.

Hint: $\lim_{n\to\infty} x_n = 0$. $y_n = \frac{x_n}{1-x_n}$

Now, $\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \left(\frac{1}{1-x_n}\right) = 1$ which is non-zero and finite.

• Prob: If $\sum x_n$ & $\sum y_n$ are conv. series of positive terms, prove that $\sum \sqrt{x_n y_n}$ is conv.

Soln: $\because \sum x_n$ & $\sum y_n$ are convergent, hence $\sum \frac{1}{2}(x_n + y_n)$ is convergent.

Now, $\sqrt{x_n y_n} \le \frac{x_n + y_n}{2}$, $\forall n \in \mathbb{N}$.

$\therefore$ By Comparison test, $\sum \sqrt{x_n y_n}$ is convergent.

(v) $\sum \frac{1}{n^p}\left(\frac{n+2}{n+3}\right)^n$, $y_n = \frac{1}{n^p}$

$\lim \frac{x_n}{y_n} = \frac{(1+\frac{2}{n})^n}{(1+\frac{3}{n})^n} = \frac{e^2}{e^3}$

(vi) $\sum (a^{1/n} - 1)$, $a \neq 1, a > 0$

$y_n = \frac{1}{n}$. $\lim \frac{x_n}{y_n} = \lim \frac{a^{1/n}-1}{1/n}$

$= \log a \neq 0$ & finite.

• Prob: Test Convergence of the series

(i) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$ (ii) $\sum_{n=2}^{\infty} \frac{1}{(\log \log n)^{\log n}}$ (vii) $\sum_{n=2}^{\infty} \cdots$

(iii) $\sum \frac{1}{n^n}$ (iv) $\sum \frac{2^n + 3^n}{6^n}$

Soln: (i) Since $\log\log n \to \infty$ as $n \to \infty$, hence $\exists\, n_0 \in \mathbb{N}$ s.t.

$\log\log n > 2$, $\forall n \ge n_0$.

$\Rightarrow$ ~~$\log\log n > \log 2, \forall n \ge n_0$~~ $\Rightarrow \log n \cdot \log\log n > 2\log n$, $\forall n \ge n_0$

$\Rightarrow \log\left((\log n)^{\log n}\right) > \log n^2$, $\forall n \ge n_0$ $\Rightarrow (\log n)^{\log n} > n^2$, $\forall n \ge n_0$

$\Rightarrow \frac{1}{(\log n)^{\log n}} < \frac{1}{n^2}$, $\forall n \ge n_0$

(ii) Similar to (i). Since $\log\log\log n \to \infty$ ....

(iii) $n^n > 2^n$, $\forall n \ge 3$. $\Rightarrow \frac{1}{n^n} < \frac{1}{2^n}$, $\forall n \ge 3$.

(iv) $\sum \frac{1}{3^n}$ and $\sum \frac{1}{2^n}$ are infinite geometric series with common ratios $\frac{1}{3}$ and $\frac{1}{2}$ resp. $\because \left|\frac{1}{3}\right| < 1$ and $\left|\frac{1}{2}\right| < 1$, hence both $\sum \frac{1}{3^n}$ and $\sum \frac{1}{2^n}$ are convergent.

$\therefore \sum \left(\frac{1}{3^n} + \frac{1}{2^n}\right)$, i.e., $\sum \left(\frac{2^n + 3^n}{6^n}\right)$ is conv.

(vii) $x_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1}$, $y_n = \frac{1}{n^{5/2}}$

(xiv) $x_n = \log(n+1) - \log n$, $y_n = \frac{1}{n}$. $\therefore \frac{x_n}{y_n} = n\log(1+\frac{1}{n}) \to 1$ as $n \to \infty$

(i) $x_n = \frac{(2n-1)\cdot 2n}{(2n+1)^2(2n+2)^2} < \frac{1}{(2n-1)2n} \le \frac{1}{n^2} = y_n$ (C)

(ii) $x_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} = y_n$ (D)

(xx) $x_n = \frac{n^n}{(n+1)^{n+1}}$, $y_n = \frac{1}{n}$

(iii) $x_n = \frac{1}{\sqrt{n(n+1)}} < \frac{1}{n} = y_n$ (D)

(iv) $x_n = \frac{n+1}{n^p}$. Take $y_n = \frac{1}{n^{p-1}}$. Then $\lim_{n\to\infty} \frac{x_n}{y_n}$

say that $\sum \frac{1}{n^{\epsilon+1}}$ is divergent ...
$\sum \frac{1}{n^{\epsilon+1}}$ is convergent, since $\epsilon+1>1$.

D'Alembert's Ratio Test:-

Let $\sum u_n$ be a series of non-negative terms. Let $\rho_1 = \overline{\lim} \frac{u_{n+1}}{u_n}$ and $\rho_2 = \underline{\lim} \frac{u_{n+1}}{u_n}$. Then (i) $\sum u_n$ converges if $\rho_1 < 1$ and (ii) $\sum u_n$ diverges if $\rho_2 > 1$.

In particular, if $\rho_1 = \rho_2 \left(= \lim_{n\to\infty} \frac{u_{n+1}}{u_n}\right)$, then (i) $\sum u_n$ converges if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$ and (ii) $\sum u_n$ diverges if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$.

No definite conclusion can be drawn if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$.

- **Probl:-** Test the convergence of the series

(i) $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} x + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} x^2 + \cdots \infty$ $(x>0)$ (ii)

(iii) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$ $(x \geq 0)$. (Conv. $0 \leq x < 1$ and div $x \geq 1$).

(iv) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ (v) $\sum \frac{n^2}{2^n}$ (vi) $\sum \frac{4^n}{n!}$ (vii) $\sum \frac{n^4}{e^{n^2}}$ (viii) $\sum \frac{x^n}{n!}$

(ix) $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \cdots$ $(x>0)$

(x) $x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \cdots$ $(x>0)$ (xiii) $\sum \frac{n!}{n^n}$

(xi) $\sum \frac{2}{n^n}$ (xii) $\sum \frac{n+2}{2^n}$ ~~(xiii) $1 + \frac{2^2}{3^2} + \frac{2^2 4^2}{3^2 5^2} + \frac{2^2 4^2 6^2}{3^2 5^2 7^2} + \cdots$~~

(xiv) $\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \frac{4^2 \cdot 5^2}{4!} + \cdots$ (xv) $\frac{1}{5} + \frac{2!}{5^2} + \frac{3!}{5^3} + \cdots$

(xvi) $\frac{1^3}{3^1} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \frac{5^3}{3^5} + \cdots$ (xvii) $\sum \frac{2^n \cdot n!}{n^n}$ (xviii) $\sum \frac{\sqrt{n}}{n^2+1} x^n$ $(x>0)$.

(xix) $\frac{1}{1.2.3} + \frac{x}{4.5.6} + \frac{x^2}{7.8.9} + \cdots$ $(x>0)$ (xx) $\frac{x^2}{2\sqrt{1}} + \frac{x^3}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots$

(xxi) $\frac{1}{2} + \frac{1.3}{2.5} + \frac{1.3.5}{2.5.8} + \cdots$ (xxii) $\frac{1}{2} + \frac{2!}{2^3} + \frac{3!}{2^5} + \frac{4!}{2^7} + \cdots$

(xxiii) $\sum \left(\frac{1}{n}\right)^{1/n}$ (xxiv) $\frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \cdots$ (xxv) $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \cdots$

(xxvi) $\sum \frac{n^2(n+1)^2}{n!}$ (xxvii) $\sum \frac{n!}{n^n}$ (xxviii) $\sum \frac{x^n}{3^n n^2}$, $x>0$.

(xxix) $1 + 2x + 3x^2 + 4x^3 + \cdots$ $(x>0)$ (xxx) $\frac{x}{1.3} + \frac{x^2}{3.5} + \frac{x^3}{5.7} + \cdots$

(xxxi) $\sum_{n=1}^{\infty} \frac{3^n - 2}{3^n + 1} x^{n-1}$, $x>0$. (xxxii) $\sum \frac{5^n}{n^2+7}$ (xxxiii) $\sum_{n=1}^{\infty} \frac{1}{5^n + k}$ $(k>0)$

(xxxiv) $\sum x^n \cos\left(\frac{1}{n}\right)$ $(x>0)$. (xxxv) $\sum \frac{n^p}{a^n}$ $(a>1)$

Soln:- (i) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$. If $x=1$, then $\frac{x_n}{x_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{(2n+1)^2+2(2n+1)+1}{(2n+1)^2} = 1+\frac{4n+3}{(2n+1)^2}$

$\therefore \frac{x_n}{x_{n+1}} = 1+\frac{4n^2+3n}{n(4n^2+4n+1)} = 1+\frac{1}{n}+\frac{-(n+1)}{n(4n^2+4n+1)} = 1+\frac{1}{n}+\frac{-\sqrt{n}(n+1)/(4n^2+4n+1)}{n^{1+1/2}}$

$\therefore$ By Gauss Test $\sum u_n$ is div for $x=1$.

(ii)

(iii) $\frac{u_{n+1}}{x_n} = \frac{x^{n+1}}{n+1}\times\frac{n}{x^n} \to x$ as $n\to\infty$. If $x=1$, then $\frac{x_n}{x_{n+1}} = \frac{n+1}{n} = 1+\frac{1}{n}+\frac{0}{n^2}$

$\therefore$ By Gauss Test, $\sum u_n$ is div, for $x=1$.

(iv) $\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}}\times\frac{2^n}{n} = \left(1+\frac{1}{n}\right)\times\frac{1}{2} \to \frac{1}{2}$ as $n\to\infty$. $\because \frac{1}{2}<1$, by D'Alembert's Radio test, $\sum u_n$ is Conv.

(vi) $\frac{x_{n+1}}{x_n} = \frac{4^{n+1}}{(n+1)!}\times\frac{n!}{4^n} = \frac{4}{n+1} \to 0$ as $n\to\infty$.

(vii) $\frac{x_{n+1}}{x_n} = \frac{(n+1)^4}{e^{(n+1)^2}}\times\frac{e^{n^2}}{n^4} = \left(1+\frac{1}{n}\right)^4\times\frac{1}{e^{2n+1}} \to 0$ as $n\to\infty$

(viii) $\frac{x_{n+1}}{x_n} = \frac{x^{n+1}}{(n+1)!}\times\frac{n!}{x^n} = \frac{x}{n+1} \to 0$ as $n\to\infty$.

(ix) $\frac{x_{n+1}}{x_n} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}\times\frac{n!}{n^n\cdot x^n} = \left(1+\frac{1}{n}\right)^n\cdot x \to ex$ as $n\to\infty$.

$\therefore$ By D'Alembert's ratio test, $\sum x_n$ is Conv. if $x<\frac{1}{e}$ and div. if $x>\frac{1}{e}$.

Let $x=\frac{1}{e}$. Then, $\frac{x_n}{x_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$.

$\therefore n\log\left(\frac{x_n}{x_{n+1}}\right) = n\log\left(\frac{e}{(1+\frac{1}{n})^n}\right) = n\left(1-n\log\left(1+\frac{1}{n}\right)\right) = \frac{1}{x}\left(1-\frac{1}{x}\log(1+x)\right)$, putting $\frac{1}{n}=x$

$= \frac{x-\log(1+x)}{x^2}$.

Now, $\lim_{x\to0}\frac{x-\log(1+x)}{x^2} = \lim_{x\to0}\frac{1-\frac{1}{1+x}}{2x}$ (By L'Hospital rule) $= \lim_{x\to0}\frac{1}{2(1+x)} = \frac{1}{2}$

$\therefore \lim_{n\to\infty} n\log\left(\frac{x_n}{x_{n+1}}\right) = \frac{1}{2}<1$.

$\therefore$ By logarithmic test, $\sum u_n$ is divergent for $x=\frac{1}{e}$.

(x) $x_n = \frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} >0, \forall n\geq1$ (neglecting the 1st term).

$\therefore \lim_{n\to\infty}\frac{x_n}{x_{n+1}} = \lim_{n\to\infty}\frac{2n+2}{2n+1}\times\frac{x^{2n+1}}{2n+1}\times\frac{2n+3}{x^{2n+3}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}\cdot\frac{1}{x^2} \to \frac{1}{x^2}$ as $n\to\infty$.

$\therefore \sum u_n$ is Conv. if $\frac{1}{x^2}>1$, i.e, $0<x<1$ and div. if $\frac{1}{x^2}<1$, i.e, $x>1$ ($\because x>0$)

If $\frac{1}{x^2}=1$, i.e, if $x=1$ ($\because x>0$), then $\frac{x_n}{x_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$

$\therefore n\left(\frac{x_n}{x_{n+1}}-1\right) = n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right) = \frac{6n^2+5}{4n^2+4n+1} \to \frac{3}{2}$ as $n\to\infty$.

$\because \frac{3}{2}>1$, by Raabe's Test $\sum u_n$ is Conv. for $x=1$.

(xi) $\frac{x_n}{x_{n+1}} = \frac{2}{n^n}\times\frac{(n+1)^{n+1}}{2} = (n+1)\cdot\left(1+\frac{1}{n}\right)^n \to \infty$ as $n\to\infty$

$\therefore$ By D'Alembert's ratio test, ~~the~~ $\sum u_n$ is Convergent.

(xii) $\frac{x_n}{x_{n+1}} = \frac{n+2}{2^n}\times\frac{2^{n+1}}{n+3} = \frac{2(n+2)}{n+3} \to 2\ (>1)$ as $n\to1$. $\therefore \sum u_n$ is Conv.

(xiii) $\frac{x_n}{x_{n+1}} = \frac{n!}{n^n}\times\frac{(n+1)^{n+1}}{(n+1)!} = \left(1+\frac{1}{n}\right)^n \to e\ (>1)$ as $n\to\infty$. $\therefore \sum u_n$ is Conv.

(xiv) $x_n = \frac{n^2(n+1)^2}{n!}$ $\therefore \frac{x_n}{x_{n+1}} = \frac{n^2(n+1)^2}{n!}\times\frac{(n+1)!}{(n+1)^2(n+2)^2} = \frac{n^2(n+1)}{(n+2)^2} \to \infty$ as $n\to\infty$

$\therefore \sum x_n$ is Conv.

(xv) $\frac{x_n}{x_{n+1}} = \frac{n!}{5^n}\times\frac{5^{n+1}}{(n+1)!} = \frac{5}{n+1} \to 0$ as $n\to\infty$. $\therefore \sum u_n$ is div.

(xvi) $\frac{x_n}{x_{n+1}} = \frac{n^3}{3^n}\times\frac{3^{n+1}}{(n+1)^3} = \frac{3n^3}{(n+1)^3} \to 3\ (>1)$ as $n\to\infty$. $\therefore \sum u_n$ is Conv.

(xvii) $\frac{x_n}{x_{n+1}} = \frac{2^n \cdot n!}{n^n} \times \frac{(n+1)^{n+1}}{2^{n+1}(n+1)!} = \frac{1}{2}\left(1+\frac{1}{n}\right)^n \to \frac{e}{2}(>1)$ as $n\to\infty$ $\therefore \Sigma x_n$ is conv.

(xviii) $\frac{x_n}{x_{n+1}} = \frac{\sqrt{n}}{n^2+1} x^n \times \frac{(n+1)^2+1}{\sqrt{n+1}\cdot x^{n+1}} = \frac{1}{x}\cdot\frac{\sqrt{n}(n^2+2n+2)}{\sqrt{n+1}(n^2+1)} \to \frac{1}{x}$ as $n\to\infty$

$\therefore \Sigma x_n$ is conv if $0<x<1$ and div if $x>1$. For $x=1$,

$$n\left(\frac{x_n}{x_{n+1}}-1\right) = n\left\{\frac{\sqrt{n}(n^2+2n+2)}{\sqrt{n+1}(n^2+1)}-1\right\} = \frac{n\left(\sqrt{n}(n^2+2n+2)-\sqrt{n+1}(n^2+1)\right)}{\sqrt{n+1}(n^2+1)} = \frac{n\{n(n^4+8n^2+4n^3+8n+4)-(n+1)(n^2+1)^2\}}{\sqrt{n+1}(n^2+1)\left(\sqrt{n}(n^2+2n+2)+\sqrt{n+1}(n^2+1)\right)}$$

$$= \frac{n(3n^4+6n^3+6n^2+3n-1)}{\sqrt{n+1}(n^2+1)\left(\sqrt{n}(n^2+2n+2)+\sqrt{n+1}(n^2+1)\right)} \to 3\,(>1) \text{ as } n\to\infty$$ $\therefore$ By Raabe's Test, Σx_n is Conv. for $x=1$.

(xix) $x_n = \frac{x^{n-1}}{(3n-2)(3n-1)3n}$ $\therefore \frac{x_n}{x_{n+1}} = \frac{x^{n-1}}{(3n-2)(3n-1)(3n)} \times \frac{(3n+1)(3n+2)(3n+3)}{x^n} \to \frac{1}{x}$ as $n\to\infty$

$\therefore \Sigma x_n$ is Conv. for $0<x<1$, div. for $x>1$. For $x=1$,

$$n\left(\frac{x_n}{x_{n+1}}-1\right) = n\left(\frac{(3n+1)(3n+2)(3n+3)}{(3n-2)(3n-1)3n}-1\right) = \frac{n\left((9n^2+9n+2)(3n+3)-(9n^2-9n+2)\cdot 3n\right)}{(3n-2)(3n-1)3n}$$

$$= \frac{n(27n^3+54n^2+33n+6-27n^3+27n^2-6n)}{(3n-2)(3n-1)3n} = \frac{n(81n^2+27n+6)}{(3n-2)(3n-1)3n} \to \frac{81}{27} = 3(>1) \text{ as } n\to\infty$$

$\therefore$ By Raabe's Test, Σx_n is Conv. for $x=1$.

(xx) $\frac{x_n}{x_{n+1}} = \frac{x^{n+1}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{n+2}} = \frac{1}{x}\cdot\frac{(n+2)}{\sqrt{n}\sqrt{n+1}} \to \frac{1}{x}$ as $n\to\infty$

For $x=1$, $n\left(\frac{x_n}{x_{n+1}}-1\right) = n\cdot\left(\frac{n+2-\sqrt{n}\sqrt{n+1}}{\sqrt{n}\sqrt{n+1}}\right) = n\left(\frac{n^2+4n+4-n^2-n}{\sqrt{n}\sqrt{n+1}(n+2+\sqrt{n}\sqrt{n+1})}\right)$

$= \frac{3n^2+4n}{\sqrt{n}\sqrt{n+1}(n+2+\sqrt{n}\sqrt{n+1})} \to \frac{3}{2}(>1)$ as $n\to\infty$. $\therefore$ By Rabbe's Test Σx_n is Conv. for $x=1$

(xxi) $\frac{x_n}{x_{n+1}} = \frac{1.3.5.\cdots(2n-1)}{2.5.8.\cdots(3n-1)} \times \frac{2.5.8.\cdots(3n-1)(3n+2)}{1.3.5.\cdots(2n-1)(2n+1)} = \frac{3n+2}{2n+1} \to \frac{3}{2}(>1)$ as $n\to\infty$

$\therefore$ By D'Alembert's Ratio test, Σx_n is Conv.

(xxii) $\frac{x_n}{x_{n+1}} = \frac{n!}{2^{2n-1}} \times \frac{2^{2n+1}}{(n+1)!} = \frac{2^2}{n+1} \to 0$ as $n\to\infty$. $\therefore$ By D'Alembert's Ratio Test Σx_n is div.

(xxiii) $n\log\left(\frac{x_n}{x_{n+1}}\right) = n\cdot\log\left(\frac{(n+1)^{\frac{1}{n+1}}}{n^{1/n}}\right) = n\left(\frac{\log(n+1)}{n+1} - \frac{\log n}{n}\right)$

$= \frac{1}{x}\left(\frac{x}{1+x}\cdot\log\left(\frac{1+x}{x}\right)+\log(x)\right)$, putting $x=\frac{1}{n}$

$= \frac{1}{1+x}\log(1+x) + \left(1-\frac{1}{1+x}\right)\log x = \frac{\log(1+x)}{1+x} + \frac{x\log x}{1+x}$

$\to \frac{\log 1}{1+0} + \frac{0}{1+0} = 0$ as $x\to 0$ ($\because \lim_{x\to 0} x\log x = 0$)

$\therefore$ By logarithmic Test Σx_n is div.

(xxiv) $\frac{x_n}{x_{n+1}} = \frac{2^n}{n^2+1} \times \frac{(n+1)^2+1}{2^{n+1}} = \frac{1}{2}\cdot\frac{n^2+2n+2}{n^2+1} \to \frac{1}{2}(<1)$ as $n\to\infty$

$\therefore$ By D'Alembert's Radio Test, Σx_n is div.

(xxv) $\frac{x_n}{x_{n+1}} = \frac{(n+1)!}{3^n} \times \frac{3^{n+1}}{(n+2)!} = \frac{3}{n+2} \to 0$ as $n\to\infty$. $\therefore$ By D'Alembert's Ratio Test, Σx_n is div.

(xxvi) $\frac{x_n}{x_{n+1}} = \frac{n^2(n+1)^2}{n!} \times \frac{(n+1)!}{(n+1)^2(n+2)^2} = \frac{n^2(n+1)}{(n+2)^2} \to \infty$ as $n\to\infty$. $\therefore$ ...... Σx_n is Conv.

(xxvii) $\frac{x_n}{x_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!} = \left(1+\frac{1}{n}\right)^n \to e(>1)$ as $n\to\infty$. $\therefore$ ...... Σx_n is Conv.

(xxviii) $\frac{x_n}{x_{n+1}} = \frac{x^n}{3^n n^2} \times \frac{3^{n+1}\cdot(n+1)^2}{x^{n+1}} = \frac{3}{x}\cdot\frac{(n+1)^2}{n^2} \to \frac{3}{x}$ as $n\to\infty$.

For $x=3$, $n\left(\frac{x_n}{x_{n+1}}-1\right) = n\left(\frac{(n+1)^2}{n^2}-1\right) = n\left(\frac{n^2+2n+1-n^2}{n^2}\right) = \frac{2n+1}{n} \to 2(>1)$ as $n\to\infty$

$\therefore$ By Raabe's Test, Σx_n is conv. for $x=3$

(xxix) $\frac{x_n}{x_{n+1}} = \frac{n x^{n-1}}{(n+1)x^n} \to \frac{1}{x}$ as $n\to\infty$.

For $x=1$, ~~$\frac{x_n}{x_{n+1}} = \frac{n}{n+1}$~~ $S_n = 1+2+\cdots+n = \frac{n(n+1)}{2} \to \infty$ as $n\to\infty$. $\therefore \Sigma x_n$ is div. for $x=1$.

(xxx) $\frac{x_n}{x_{n+1}} = \frac{x^n}{(2n-1)(2n+1)} \times \frac{(2n+1)(2n+3)}{x^{n+1}} = \frac{1}{x}\frac{2n+3}{2n-1} \to \frac{1}{x}$ as $n\to\infty$.

For $x=1$, $n\left(\frac{x_n}{x_{n+1}}-1\right) = n\left(\frac{2n+3}{2n-1}-1\right) = \frac{4n}{2n-1} \to \frac{4}{2} = 2\,(>1)$ as $n\to\infty$.

$\therefore$ By Raabe's Test, Σu_n is conv. for $x=1$.

(xxxi) $\frac{x_n}{x_{n+1}} = \frac{3^n-2}{3^n+1}\cdot x^{n-1} \times \frac{3^{n+1}+1}{3^{n+1}-2} \times \frac{1}{x^n} \to \frac{1}{x}$ as $n\to\infty$.

For $x=1$, $n\left(\frac{x_n}{x_{n+1}}-1\right) = n\left(\frac{(3^n-2)(3.3^n+1)-(3^n+1)(3.3^n-2)}{(3^n+1)(3.3^n-2)}\right)$

$= n\left(\frac{3.3^{2n}+3^n-6.3^n-2-3.3^{2n}+2.3^n-3.3^n+2}{(3^n+1)(3.3^n-2)}\right) = \frac{-6.3^n.n}{(3^n+1)(3.3^n-2)}$

$= \frac{-6.\,n/3^n}{(1+1/3^n)(3-2/3^n)} \to 0$ as $n\to\infty$.

$\therefore$ By Raabe's Test, Σu_n is div. for $x=1$.

(xxxii) $\frac{x_n}{x_{n+1}} = \frac{5^n}{n^2+7} \times \frac{(n+1)^2+7}{5^{n+1}} \to \frac{1}{5}\,(<1)$ as $n\to\infty$ $\therefore$ By D'Alembert's Ratio Test Σu_n is div.

(xxxiii) $\frac{u_n}{u_{n+1}} = \frac{1}{5^n+k} \times (5^{n+1}+k) \to 5\,(>1)$ as $n\to\infty$. $\therefore$ " " " " " conv.

(xxxiv) $\frac{x_n}{x_{n+1}} = \frac{x^n \cos\left(\frac{1}{n}\right)}{x^{n+1}\cos\left(\frac{1}{n+1}\right)} \to \frac{1}{x}$ as $n\to\infty$.

For $x=1$, $n\left(\frac{x_n}{x_{n+1}}-1\right) = \frac{n\left(\cos\frac{1}{n}-\cos\left(\frac{1}{n+1}\right)\right)}{\cos\left(\frac{1}{n+1}\right)} = \frac{2n.\sin\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n}\right)\sin\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n}\right)}{\cos\left(\frac{1}{n+1}\right)}$

$= \frac{-2n\sin\left(\frac{1}{2n(n+1)}\right)\sin\left(\frac{2n+1}{2n(n+1)}\right)}{\cos\left(\frac{1}{n+1}\right)} = \frac{-\frac{2}{x}\cdot\sin\left(\frac{x^2}{2(1+x)}\right)\sin\left(\frac{x(2+x)}{2(1+x)}\right)}{\cos\left(\frac{x}{1+x}\right)}$ $\left(x=\frac{1}{n}\right)$

$= \frac{-\frac{\sin\left(\frac{x^2}{2(1+x)}\right)}{\frac{x^2}{2(1+x)}} \times \frac{\sin\left(\frac{x(2+x)}{2(1+x)}\right)}{\frac{x(2+x)}{2(1+x)}} \times \frac{2}{x} \times \frac{x^2}{2(1+x)}\cdot\frac{x(2+x)}{2(1+x)}}{\cos\left(\frac{x}{1+x}\right)} \to 0$ as $x\to 0$

$\therefore$ By Raabe's Test, Σu_n is div. for $x=1$.

(xxxv) $\frac{x_n}{x_{n+1}} = \frac{n^p}{a^n} \times \frac{a^{n+1}}{(n+1)^p} = a\cdot\frac{n^p}{(n+1)^p} \to a$ as $n\to\infty$.

$\because a>1$, by D'Alembert's Ratio test Σu_n is conv.

~~• Some More Problems on Cauchy's root Test~~

Logarithmic Test:- Let $\sum u_n$ be a series of positive terms s.t. $\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = l$.
Then (i) $\sum u_n$ is conv. if $l>1$, and (ii) $\sum u_n$ is div. if $l<1$
If $l=1$, then this test fails.

Gauss Test:- If $\sum u_n$ be a series of positive terms s.t. $\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+\frac{\gamma_n}{n^{1+\delta}}$
where $\delta>0$ and $\{\gamma_n\}$ is a bounded seqn, then
(i) $\sum u_n$ is conv. if $\lambda>1$, and (ii) $\sum u_n$ is div if $\lambda\le 1$.

Working rule:- (a) Comparison test is applied only when x_n does not contain any power of x involving n and when x_n does not contain factorials involving n.

(b) Cauchy's root test is applied when x_n contains the n^{th} power of itself as a whole.

(c) When the series involves increasing powers of x, we start directly with D'Alembert's ratio test.

(d) When Cauchy's root test, D'Alembert's ratio test fails then comparison test may be applied. Raabe's Test or Gauss Test may be applied also, provided $\frac{u_n}{u_{n+1}}$ does not contain e.

(e) If $\frac{u_n}{u_{n+1}}$ contains e, apply logarithmic test

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Ex:- $1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7}+\cdots$ (neglecting the 1st term)

$$\frac{u_n}{u_{n+1}} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{1}{2n+1}\times\frac{2\cdot4\cdot6\cdots(2n)(2n+2)}{1\cdot3\cdot5\cdots(2n-1)(2n+1)}\times(2n+3)$$
$$= \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} \to 1 \text{ as } n\to\infty.$$

Now, $n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{4n^2+10n+6-4n^2-4n-1}{(2n+1)^2}\right) = \frac{6n^2+5n}{(2n+1)^2} \to \frac{6}{4}=\frac{3}{2}(>1)$ as $n\to\infty$

$\therefore$ By Raabe's Test, $\sum u_n$ is Conv.

Ex:- $\frac{1}{2}+\frac{1\cdot3}{2\cdot4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}+\cdots$

$$\frac{u_n}{u_{n+1}} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdots(2n)}\times\frac{2\cdot4\cdots(2n)(2n+2)}{1\cdot3\cdot5\cdots(2n-1)(2n+1)} = \frac{2n+2}{2n+1} \to 1 \text{ as } n\to\infty.$$

$\therefore n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{2n+2-2n-1}{2n+1}\right) = \frac{n}{2n+1} \to \frac{1}{2}\ (<1)$ as $n\to\infty$.

$\therefore$ By Raabe's Test, $\sum u_n$ is div.

• Ex: (i) ~~$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\cdots (x>0)$~~

(ii) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^p}$

~~(i) $\frac{u_n}{u_{n+1}} = \frac{1}{(1+\frac{1}{n})^n}\cdot\frac{1}{x} \to \frac{1}{ex}$ as $n\to\infty$. $\therefore$ Conv. if $x<\frac{1}{e}$ and div. if $x>\frac{1}{e}$ (by D'Alembert's Ratio Test)~~

~~For $x=\frac{1}{e}$,~~ $\frac{u_n}{u_{n+1}} = \frac{e}{(1+\frac{1}{n})^n}$

$$\frac{u_n}{u_{n+1}} = \left(\frac{\log(n+1)}{\log n}\right)^p = \left(\frac{\log\left(\frac{x+1}{x}\right)}{-\log x}\right)^p \left(x=\frac{1}{n}\right) = \left(\frac{\log(1+x)-\log x}{-\log x}\right)^p$$
$$= \left(1-\frac{\log(1+x)}{\log x}\right)^p \to (1-0)^p = 1 \text{ as } x\to 0, \text{ i.e., } n\to\infty$$

Now, $n \log\left(\frac{u_n}{u_{n+1}}\right) = n \cdot p \log\left(\frac{\log(n+1)}{\log n}\right) = \cancel{p \cdot \frac{\log(\log(n+1)) - \log(\log n)}{}}$

$$= \frac{p}{n} \cdot \log\left(1 - \frac{\log(1+n)}{\log n}\right)$$

$$= \frac{p \cdot \log\left(1 - \frac{\log(1+n)}{\log n}\right)}{\frac{\log(1+n)}{\log n}} \times \frac{\log(1+n)}{n \log n} \to p \times 1 \times 1 \times 0$$

as $n \to 0$ $\left(\because \frac{\log(1+n)}{\log n} \text{ as } n \to 0\right)$

~~Now,~~ $\cancel{\lim_{y \to 0} \frac{\log(1+y)}{y}}$

$\therefore n \log\left(\frac{u_n}{u_{n+1}}\right) \to 0$ as $n \to \infty$.

$\therefore$ By logarithmic test $\sum u_n$ is div., $\forall$ real value p.

Cauchy's Condensation Test:- If $\phi(n) > 0$, $\forall n \in \mathbb{N}$ and the seqn. $\{\phi(n)\}$ is m.d., then the two series $\sum_{n=1}^{\infty} \phi(n)$ and $\sum_{n=1}^{\infty} h^n \phi(h^n)$ $(h \in \mathbb{N} \setminus \{1\})$ are either both convergent or both divergent.

• **Ex:-** Test convergence of the series

(i) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (ii) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (iii) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$.

(i) If $p = 0$, then $S_n = n \to \infty$ as $n \to \infty$ and so in this case $\sum u_n$ is div.
If $p < 0$, then $\sum u_n$ is obviously div.
Let $p > 0$. Take $\phi(n) = \frac{1}{n^p}$, $\forall n \in \mathbb{N}$. Then the seqn. $\{\phi(n)\}$ is m.d. and $\phi(n) > 0$, $\forall n \in \mathbb{N}$.
Then by Cauchy's Condensation Test both the series $\sum_{n=1}^{\infty} \phi(n)$ and $\sum_{n=1}^{\infty} 2^n \phi(2^n)$ are either conv. or div.
Now $\sum_{n=1}^{\infty} 2^n \phi(2^n) = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(2^{p-1})^n}$ —— (1)
If $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{(2^{p-1})^n}$ is an infinite geometric series with common ratio $\frac{1}{2^{p-1}}$ and $\left|\frac{1}{2^{p-1}}\right| < 1$ and so the series (1) is conv. and hence the given series is conv., in this case.
If $p = 1$, then for the series (1), $S_n = n \to \infty$ as $n \to \infty$ and hence the series (1) is div. ~~in this case~~ and hence the given series is div., in this case.
If $0 < p < 1$, then from (1), we get
$\sum_{n=1}^{\infty} 2^n \phi(2^n) = \sum_{n=1}^{\infty} (2^{1-p})^n$ which is an infinite geometric series with common ratio 2^{1-p} and $|2^{1-p}| > 1$.
$\therefore \sum_{n=1}^{\infty} 2^n \phi(2^n)$ is div. and hence the given series is div., in this case.
$\therefore$ The given series is conv. for $p > 1$ and div. for $p \leq 1$.

(ii) If $p < 0$, then $\forall n \geq 2$, $x_n = \frac{1}{n(\log n)^p} = \frac{(\log n)^{-p}}{n} \geq \frac{1}{n} = y_n$ (say).
Now $\sum_{n=2}^{\infty} \frac{1}{n}$ is known to be div. Hence by Comparison test, $\sum u_n$ is div., in this case.
Let $p \geq 0$. Take $\phi(n) = \frac{1}{n(\log n)^p}$, $\forall n \geq 2$. Then clearly, $\phi(n) > 0$, $\forall n \geq 2$ and $\{\phi(n)\}$ is m.d.
$\therefore$ By Cauchy's Condensation Test both the series $\sum_{n=2}^{\infty} \phi(n)$ and $\sum_{n=2}^{\infty} 3^n \phi(3^n)$ are either conv. or div.
Now, $3^n \phi(3^n) = 3^n \cdot \frac{1}{3^n (\log 3^n)^p} = \frac{1}{n^p (\log 3)^p}$
$\therefore \sum_{n=2}^{\infty} 3^n \phi(3^n) = \frac{1}{(\log 3)^p} \sum_{n=2}^{\infty} \frac{1}{n^p}$ which is conv. for $p > 1$ and div. for $0 \leq p \leq 1$.
$\therefore$ The given series is conv. for $p > 1$ and div. for $0 \leq p \leq 1$.
Hence the given series is conv. for $p > 1$ and div. for $p \leq 1$.

(iii) $p \le 0 \to$ div. Let $p>0$. $\phi(n) = \frac{1}{(\log n)^p} > 0, \forall n \ge 2$ and $\{\phi(n)\}$ is m.d.

Now $\sum_{n=2}^{\infty} 3^n \phi(3^n) = \sum_{n=2}^{\infty} 3^n \cdot \frac{1}{(\log 3^n)^p} = \sum_{n=2}^{\infty} \frac{3^n}{(\log 3)^p \cdot n^p} = \frac{1}{(\log 3)^p} \cdot \sum_{n=2}^{\infty} \frac{3^n}{n^p} = \frac{1}{(\log 3)^p} \sum y_n$ —— (1)

where $y_n = \frac{3^n}{n^p}, \forall n \ge 2$.

Now, $\frac{y_n}{y_{n+1}} = \frac{3^n}{n^p} \times \frac{(n+1)^p}{3^{n+1}} = \frac{1}{3} \cdot \left(1+\frac{1}{n}\right)^p \to \frac{1}{3} (<1)$ as $n \to \infty$.

$\therefore$ By D'Alembert's Ratio test, $\sum_{n=2}^{\infty} y_n$ is div., $\forall p > 0$.

$\therefore$ ~~By Cauchy's~~ $\sum_{n=2}^{\infty} 3^n \phi(3^n)$ is div. $\forall p>0$ (by (1))

$\therefore$ By Cauchy's Condensation Test, the given series is div. $\forall p > 0$.

$\therefore$ The given series is div. for all p.

Cauchy's Integral Test :-

If for $x \ge 1$, $f(x)$ is a non-negative m.d. integrable function of x s.t. $f(n) = x_n, \forall n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} x_n$ and the integral $\int_1^{\infty} f(x)dx$ converge or diverge together.

Ex:- Test the convergence of the series

(i) $\sum \frac{1}{n^2+1}$ (ii) $\sum \frac{1}{n^{2/3}}$ (iii) $\sum \frac{1}{n\sqrt{n^2-1}}$

Soln:- (i) $f(x) = \frac{1}{x^2+1} > 0, \forall x \ge 1$. Then $f(n) = \frac{1}{n^2+1} = x_n$ (say), $\forall n \in \mathbb{N}$.

$f(x) - f(y) = \frac{y^2-x^2}{(x^2+1)(y^2+1)} = \frac{(y-x)(y+x)}{(x^2+1)(y^2+1)} > 0, \forall y(\ge 1) > x(\ge 1)$.

$\therefore f$ is m.d., $\forall x \ge 1$.

Clearly f is an integrable function.

$\therefore$ By Integral Test, the series $\sum x_n$ and the integral $\int_1^{\infty} f(x)dx$ converge or diverge together.

Now, $\int_1^{\infty} f(x)dx = \lim_{B \to \infty} \int_1^{B} \frac{dx}{x^2+1} = \lim_{B\to\infty} [\tan^{-1}B - \tan^{-1}1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

$\therefore \int_1^{\infty} f(x)dx$ is convergent and converges to $\frac{\pi}{4}$.

$\therefore$ The given series is convergent.

■ **Alternating Series**:- An infinite series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$ or $\sum_{n=1}^{\infty}(-1)^n x_n$, where $x_n>0, \forall n\in\mathbb{N}$, is called an alternating series.

For example, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is an alternating series.

■ **Leibnitz's Test of Convergence for an alternating series**:-

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$ converges if
(i) $\{x_n\}$ is m.d. and (ii) $\lim_{n\to\infty} x_n = 0$ as $n\to\infty$.

• Note:- Converse of Leibnitz's Test of Convergence for an alternating series is not true. For example, consider the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$, where $x_n = \begin{cases}\frac{1}{n^2}, \text{ if } n \text{ be even}\\ \frac{1}{n^{10}}, \text{ if } n \text{ be odd.}\end{cases}$

Then $\sum_{n=1}^{\infty}|(-1)^{n-1}x_n| = \sum_{n=1}^{\infty}x_n$ is convergent, since $x_n \le \frac{1}{n^2} = y_n(\text{say}), \forall n\ge 1$.
and $\sum y_n$ is convergent, by comparison test.

Now, $\forall n\in\mathbb{N}$, $(-1)^{n-1}x_n \le |(-1)^{n-1}x_n| \Rightarrow \sum_{n=1}^{\infty}(-1)^{n}x_n$ is convergent, by comparison test.

Hence $\lim_{n\to\infty}x_n = 0$, but $\{x_n\}$ is not m.d., for $x_1 = 1, x_2 = \frac{1}{4}, x_3 = \frac{1}{3^{10}}, x_4 = \frac{1}{16}$, so that $x_1 > x_2 > x_3 < x_4$.

• Ex:- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$.

Soln:- The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$, where $x_n = \frac{1}{n^2} > 0, \forall n\in\mathbb{N}$. ∴ The given series is an alternating series.

Now, $\lim_{n\to\infty}x_n = \lim_{n\to\infty}\frac{1}{n^2} = 0$.

Also, $x_n - x_{n+1} = \frac{1}{n^2} - \frac{1}{(n+1)^2} = \frac{2n+1}{n^2(n+1)^2} > 0, \forall n\in\mathbb{N}$

∴ $\{x_n\}$ is m.d.

∴ By Leibnitz's Test of convergence for alternating series, the given series is convergent.

• **Defn:-(Mixed Series)** An infinite series $\sum_{n=1}^{\infty}x_n$ having infinite number of positive terms and infinite number of negative terms is called a mixed series.

• **Defn:-** (i) An infinite series $\sum_{n=1}^{\infty}x_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty}|x_n|$ is convergent.

(ii) An infinite series $\sum_{n=1}^{\infty}x_n$ is said to be conditionally convergent if the series $\sum_{n=1}^{\infty}x_n$ converges but not absolutely (i.e., the series $\sum_{n=1}^{\infty}|x_n|$ does not converge).

• Note:- It is clear that there is no difference between conditionally convergence and absolutely convergence of an infinite series $\sum x_n$ of non-negative terms, since $|x_n| = x_n, \forall n\in\mathbb{N}$.

• **Th:-** If a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent then $\sum_{n=1}^{\infty} x_n$ is convergent, but not conversely.

Pf:- Let $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. Then the series $\sum_{n=1}^{\infty} |x_n|$ is convergent. Therefore for any given $\varepsilon>0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$\big|\,|x_{n+1}|+|x_{n+2}|+\cdots+|x_{n+p}|\,\big| < \varepsilon,\ \forall n\geq n_0,\ \forall p\geq 1. \quad \text{--- (1)}$$

Now, $|x_{n+1}+x_{n+2}+\cdots+x_{n+p}| \leq |x_{n+1}|+|x_{n+2}|+\cdots+|x_{n+p}|$

$$= \big|\,|x_{n+1}|+|x_{n+2}|+\cdots+|x_{n+p}|\,\big| < \varepsilon,\ \forall n\geq n_0, \forall p\geq 1 \ \text{(by (1))}.$$

$\therefore$ By Cauchy's criterion for convergence of an infinite series, the series $\sum_{n=1}^{\infty} x_n$ is convergent.

To disprove the converse part, consider the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$, where $x_n=\frac{1}{n}, \forall n\in\mathbb{N}$. Clearly $\{x_n\}$ is a m.d. seqn. and $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{1}{n}=0$. Therefore, by Leibnitz's Test of convergence of an alternating series, $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$ is convergent.

Now, $\sum_{n=1}^{\infty}|(-1)^{n-1}x_n| = \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty}\frac{1}{n}$ is a p-series with $p=1$ so that $\sum_{n=1}^{\infty}|(-1)^{n-1}x_n|$ is not convergent.

• **Th:-** (a) If a series is absolutely convergent, then the series formed by its positive terms alone is convergent, and the series formed by its negative terms alone is convergent.

(b) If a series is conditionally convergent, then the series formed by its positive terms alone is divergent and the series formed by its negative terms alone is divergent.

Ex: Test the convergence and absolute convergence of the series

(i) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$ (ii) $\frac{1}{1\cdot3}-\frac{1}{2\cdot4}+\frac{1}{3\cdot5}-\frac{1}{4\cdot6}+\cdots$

(iii) $\sum_{n=2}^{\infty}(-1)^n\cdot\frac{1}{\log n}$ (iv) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p}$ (v) $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots$

(vi) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ (vii) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\sqrt{n}}$ (viii) $1-\frac{1+2}{2^3}+\frac{1+2+3}{3^3}-\cdots$

(ix) $\sum_{n=1}^{\infty}(-1)^{n-1}\sin\left(\frac{1}{n}\right)$ (x) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2n-1}$ (xi) $\sum\frac{(-1)^{n-1}n}{10n-1}$ (xii) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{5^n}$

(xiii) $\frac{1}{\sqrt{2}+1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}+1}-\frac{1}{\sqrt{5}+1}+\cdots$ (xiv) $\sum_{n=1}^{\infty}\frac{(-1)^n(n+2)}{2^n+5}$

(xv) $1-\frac{1}{4\cdot3}+\frac{1}{4^2\cdot5}-\frac{1}{4^3\cdot7}+\cdots$ (xvi) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}$

(xvii) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n^2+1}-n\right)$ (xviii) $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\log n)^p}$ $(p>0)$.

(xix) $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}\right)$. (xx) $\sum_{n=1}^{\infty}\frac{(-1)^n(n+5)}{n(n+1)}$

~~(xxi) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$.~~

Solns (i) $\sum(-1)^{n-1}x_n$, where $x_n=\frac{1}{2n-1}, \forall n\in\mathbb{N}$. $\to$ Conv. by L. Test.

Now $\sum|(-1)^{n-1}x_n| = \sum x_n = \sum\frac{1}{2n-1} \to$ div., by Comp. Test $y_n=\frac{1}{2n}$.

$\therefore$ The given series is conditionally conv.

(ii) $\sum(-1)^{n-1}x_n$, where $x_n=\frac{1}{n(n+2)}$.

$\sum x_n$ is conv. by Comp Test with $y_n=\frac{1}{n^2}$.

$\therefore$ The given series is absolutely conv. and hence conv.

(iii) By Leibnitz's Test, the series is conv.
But $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is div. by Comp. Test with $y_n = \frac{1}{n} \le \frac{1}{\log n}, \forall n \ge 2$.

(iv) By Leibnitz's Test, the series is conv. (For m.d. of $\{\frac{1}{n^p}\}$, consider $f(x) = \frac{1}{x^p}, x>0$. Then $f'(x) = \frac{-p}{x^{p+1}} < 0, \forall x>0$ ($\because p>0$) $\therefore f$ is m.d.).
Now $\sum \frac{1}{n^p}$ is conv. for $p>1$ & div. for $p \le 1$.

(v) $\sum (-1)^{n-1} x_n$, where $x_n = \frac{1}{(2n-1)!}$.
~~$x_n - x_{n+1} = \frac{1}{(2n-1)!} - \frac{1}{(2n+1)!} = \frac{(2n+1)!(2n(2n+1)-1)}{(\ )(\ )} > 0$.~~
~~$\therefore \{x_n\}$ is m.d. Clearly $x_n \to 0$ as $n \to \infty$.~~
So $\frac{x_n}{x_{n+1}} = \frac{(2n+1)!}{(2n-1)!} = 2n(2n+1) \to \infty$ as $n \to \infty$.
$\therefore \sum x_n$ is conv (by ratio Test) so that the given series is absolutely conv. and hence it is conv.

(vi) Cond. Conv.

(vii) Abs. Conv.

(viii) $\sum (-1)^{n-1} x_n$, where $x_n = \frac{1+2+\cdots+n}{n^3} = \frac{n(n+1)}{2n^3} = \frac{n+1}{2n^2}$.
$\therefore$ Conditionally Conv.

(ix) $x_n = \sin\frac{1}{n}$.
Now, $\forall n \in \mathbb{N}$ ~~neglecting the 1st term, we can take $n \ge 2$~~
~~so that~~ $0 < \frac{1}{n} < \frac{\pi}{2}$ so that $x_n > 0, \forall n \in \mathbb{N}$.
In the 1st quadrant, sine function is m.i. so that
$x_n = \sin\frac{1}{n} \ge \sin\frac{1}{n+1} = x_{n+1}, \forall n \in \mathbb{N}$.
$\therefore \{x_n\}$ is m.d. Again, $x_n \to 0$ as $n \to \infty$.
$\therefore$ By Leibnitz's Condition, the given series is conv.
Consider $\sum |(-1)^{n-1} \sin\frac{1}{n}| = \sum \sin\frac{1}{n} \to$ div. by Ratio Test ($y_n = \frac{1}{n}$).
$\therefore$ The series is Conditionally Conv.

(x) $x_n - x_{n+1} = \frac{n}{2n-1} - \frac{n+1}{2n+1} = \frac{2n^2+n-2n^2+n-2n+1}{(2n-1)(2n+1)} = \frac{1}{(\ )(\ )} > 0$
$\therefore \{x_n\}$ is m.d.
Now, $\lim_{n\to\infty} x_n = \frac{1}{2} \ne 0$.
$\therefore$ The given series is not conv.
$\because$ The given series is not conv., hence it is not abs. conv., for then the given series is conv.

(xii) $x_n = \frac{n}{5^n}$. $\therefore \frac{x_n}{x_{n+1}} = \frac{n}{5^n} \times \frac{5^{n+1}}{n+1} \to 5 (>1)$ as $n \to \infty$.
$\therefore$ The series is abs. conv. and hence is conv.

(xiii) Cond. Conv.

(xiv) $x_n = \frac{n+2}{2^n+5}$ $\therefore \frac{x_n}{x_{n+1}} = \frac{n+2}{2^n+5} \times \frac{2^{n+1}+5}{n+3} = \frac{n+2}{n+3} \times \frac{2^{n+1}+5}{2^n+5} = \frac{1+\frac{2}{n}}{1+\frac{3}{n}} \times \frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}}$
$\to 2 (>1)$ as $n \to \infty$
$\therefore$ Abs. conv.

(xv) $x_n = \frac{1}{4^{n-1}(2n-1)}$. $\therefore \frac{x_n}{x_{n+1}} = \frac{4^n(2n+1)}{4^{n-1}(2n-1)} \to 4 (>1)$ as $n \to \infty$
$\therefore$ Abs. conv.

(xvi) Cond. Conv.

(xvii) $\sum(-1)^{n+1}(\sqrt{n^2+1}-n) = \sum(-1)^{n+1}\frac{1}{\sqrt{n^2+1}+n}$ $\therefore$ Cond. Conv.

(xviii) $x_n = \frac{1}{n(\log n)^p}$. $\sum x_n$ is Conv. for $p>1$ and div. for $0<p\le 1$ by Cauchy's Condensation Test.

(xx) Cond. Conv. (xix) Abs. Conv, since $\sum\frac{1}{n^2}$ and $\sum\frac{1}{(n+1)^2}$ are Conv.

• Ex:- Test the Conv. of the series

(i) $\sum_{n=2}^{\infty}(-1)^n.\frac{\log n}{n^2}$ (ii) $\sum_{n=2}^{\infty}(-1)^n.\frac{\log n}{n^3}$.

Soln:- (i) $f(x) = \frac{\log x}{x^2}$, $x\ge 2$. Find $f'(x)$, Then $\{x_n\}$ is m.d. $\therefore$ Conv. by L. Test

(ii) Same process.

• Ex:- Test the Convergence of the series

$\sum_{n=1}^{\infty}(-1)^n.\frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$.

Soln: $x_n - x_{n+1} = \frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \frac{1}{(n+1)}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)$

$= \frac{(n+1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - n\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)}{n(n+1)}$

$= \frac{(n+1-n)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{n}{n+1}}{n(n+1)} = \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\left(\frac{1}{n+1}+\frac{1}{n+1}+\cdots+\frac{1}{n+1}\right)}{n(n+1)} > 0.$

$\because \lim_{n\to\infty}\frac{1}{n}=0$, by Cauchy's 1st limit theorem,

$\lim_{n\to\infty}x_n = 0$ $\therefore$ By L-Test, the given series is Conv.

• Ex:- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos^2(nx)}{n\sqrt{n}}$.

$x_n = \frac{\cos^2(nx)}{n\sqrt{n}} \le \frac{1}{n^{3/2}} = y_n$.

$\therefore$ Abs. Conv.

• Ex:- Test for absolute Conv. of the series:

(i) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n!}$ (ii) $\sum_{n=1}^{\infty}(-1)^n.\frac{n^{100}}{(2n)!}$ (iii) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^3}{(n+1)!}$

(i) $\frac{x_n}{x_{n+1}} = \frac{2^n}{n!}\times\frac{(n+1)!}{2^{n+1}} \to\infty$ as $n\to\infty$. $\therefore$ Abs. Conv.

(ii) $\frac{x_n}{x_{n+1}} = \frac{n^{100}}{(2n)!}\times\frac{(2n+2)!}{(n+1)^{100}} \to\infty$ as $n\to\infty$. $\therefore$ Abs. Conv.

(iii) Same process

• Ex:- Test the Convergence of the series

$1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$ for all values of x.

Soln:- $x_n = \frac{x^{n-1}}{(n-1)!}$, $\forall n\in\mathbb{N}$.

$\therefore \frac{|x_n|}{|x_{n+1}|} = \frac{|x|^{n-1}}{(n-1)!}\times\frac{n!}{|x|^n} = \frac{1}{|x|}.n \to\infty$ as $n\to\infty$, $\forall x\ne 0$.

$\therefore \sum|x_n|$ is Conv. so that $\sum x_n$ is abs. Conv. for $x\ne 0$. $\therefore \sum x_n$ is Conv. for $x\ne 0$

For $x=0$, the series is Conv.

∴ The Series is Conv. for all x.

• **Ex:-** Test the Conv. and abs. Conv. of the series

(i) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ (ii) $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n \cdot n^2}$

Soln:- (i) ~~$\Sigma (-1)^{n-1} x_n$, where $x_n =$~~

Σu_n, where $u_n = (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$

$\therefore \left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{x^{2n-1}}{2n-1} \times \frac{2n+1}{x^{2n+1}}\right| = \frac{1}{x^2}\left(\frac{2n+1}{2n-1}\right) \to \frac{1}{x^2}$ as $n \to \infty$.

∴ Abs. Conv. for $\frac{1}{x^2} > 1$, i.e, $x^2 < 1$, i.e, $-1 < x < 1$.

For $x = 1$ → Alternating series → Conv.

For $x = -1$ → ,, ,, → Conv.

(ii) $\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{(x+1)^n}{2^n \cdot n^2} \times \frac{2^{n+1}(n+1)^2}{(x+1)^{n+1}}\right| = \frac{2 \cdot (n+1)^2}{n^2} \cdot \frac{1}{|x+1|} \to \frac{2}{|x+1|}$ as $n \to \infty$

∴ Abs. conv. for $\frac{2}{|x+1|} > 1$, i.e, $|x+1| < 2$, i.e, $-2 < x+1 < 2$, i.e, $-3 < x < 1$.

For $x = 1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ → abs. Conv.

For $x = -3$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ → Conv.

■ Rearrangement of Series:- A series Σy_n is said to be a rearrangement of the series Σx_n if $y_1+y_2+\cdots+y_n+\cdots$ with its terms in a different order, is the series $x_1+x_2+\cdots+x_n+\cdots\cdots$

i.e, every y_n is in somewhere in Σx_n and every x_n is in somewhere in Σy_n.

For example, the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ is an rearrangement of $-\frac{1}{2}+1-\frac{1}{4}+\frac{1}{3}-\cdots\cdots$

• Th:- If a series Σx_n is absolutely convergent, then any rearrangement of Σx_n also converges to the same sum as that of Σx_n.

• Probl:- Show that the two series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\cdots$ and $1+\frac{1}{3^2}-\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{4^2}+\cdots$ converge to the same sum.

Soln:- 1st series is absolutely convergent and the 2nd series is a rearrangement series of the 1st series.

• Note:- Consider the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots$ which is clearly conditionally convergent. Let it converge to s.

Now, $s=\lim_{n\to\infty} S_n=\lim_{n\to\infty} S_{2n}=\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right)$

$=\lim_{n\to\infty}\left(\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)\right)$

$=\lim_{n\to\infty}\left(\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right)$ —— (1)

Now, we know that the seqn. $\{\gamma_n\}$, where $\gamma_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n,\ \forall n\in\mathbb{N}$, converges to the Euler's constant γ. —— (2)

Now, from (1), we get

$s=\lim_{n\to\infty}\left(\gamma_{2n}+\log(2n)-(\gamma_n+\log n)\right)$ (by (2)).

$=\lim_{n\to\infty}\left(\gamma_{2n}+\log 2+\log n-\gamma_n-\log n\right)=\gamma+\log 2-\gamma=\log 2.$

∴ The series converges to $\log 2$.

Now consider the rearrangement series $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots\cdots$ of the given series.

Let σ_n be the nth partial sum of this series, for any $n\in\mathbb{N}$.

Then $\sigma_{3n} = \left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)$

$= \left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{4n-3}+\frac{1}{4n-1}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$

$= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots+\frac{1}{4n-3}+\frac{1}{4n-2}+\frac{1}{4n-1}+\frac{1}{4n}\right)$

$-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\frac{1}{2}\left(\log n+\gamma_n\right)$

$= \log(4n)+\gamma_{4n}-\frac{1}{2}\left(\log(2n)+\gamma_{2n}\right)-\frac{1}{2}\left(\log n+\gamma_n\right)$

$= 2\log 2+\log n+\gamma_{4n}-\frac{1}{2}\log 2-\frac{1}{2}\log n-\frac{1}{2}\gamma_{2n}-\frac{1}{2}\log n-\frac{1}{2}\gamma_n.$

$= \frac{3}{2}\log 2+\gamma_{4n}-\frac{1}{2}\gamma_{2n}-\frac{1}{2}\gamma_n \to \frac{3}{2}\log 2+\gamma-\frac{1}{2}\gamma-\frac{1}{2}\gamma$

$= \frac{3}{2}\log 2$ as $n\to\infty$.

Now, $\sigma_{3n+1} = \sigma_{3n}+\frac{1}{4n+1} \to \frac{3}{2}\log 2$ as $n\to\infty$

And $\sigma_{3n+2} = \sigma_{3n}+\frac{1}{4n+1}+\frac{1}{4n+3} \to \frac{3}{2}\log 2$ as $n\to\infty$.

$\therefore \lim_{n\to\infty}\sigma_n = \frac{3}{2}\log 2.$

- Riemann's Th:- If $\sum x_n$ is conditionally convergent, then for any real number s, $\exists$ a rearrangement of $\sum x_n$ converging to s. Also $\exists$ a rearrangement of $\sum x_n$ diverging to $\pm\infty$. (Book of Maity & Ghosh - page-282)

- Probl:- Assuming $\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots\infty$, show that

(i) $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}\cdots\cdots = \frac{1}{2}\log 2$

(ii) $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}+\frac{1}{5}\cdots\cdots = 0$

(iii) $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}-\frac{1}{8}+\cdots\cdots = \frac{1}{2}\log 6$

(iv) $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\cdots$ diverges.

Soln:- (i) $\sigma_{3n} =$

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Infinite Series.

Study Materials

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