Unit-2

Lecture Note - 1

Homogeneous Linear Systems with constant coefficients :-

A system of n linear differential equation is in normal form if it is expressed as

$$\dot{x}(t) = A(t)x(t) + f(t), \qquad (1)$$

where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$ and

$A(t) = [a_{ij}(t)]$ is an $n \times n$ matrix.

The system (1) is said to be homogeneous when $f(t)=0$, otherwise the system is nonhomogeneous.

The system (1) is said to be homogeneous linear system with constant coefficients if $f(t)=0$ and $A(t) = A = [a_{ij}]_{n\times n}$ i.e each element of A is constant.

An nth-order linear differential equation

$$y^{n}(t) + p_{n-1}(t)\, y^{n-1}(t) + \cdots + p_0(t)\, y(t) = g(t) \qquad (2)$$

can be written as a first-order system in normal form using substitution $x_1(t) = y(t)$, $x_2(t) = y'(t)$, ...

$\therefore x_n(t) = y^{n-1}(t)$. Equation (2) is equivalent to

$$\dot{x}(t) = A(t)\,x(t) + f(t),$$

where $x(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $f(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{pmatrix}$, and

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-2}(t) & -p_{n-1}(t) \end{bmatrix}.$$

⊛ Using matrices to solve linear systems

In this method we consider the homogeneous system of linear simultaneous differential equations of the following form:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a x_1 + b x_2 \\ \frac{dx_2}{dt} &= c x_1 + d x_2 \end{aligned}\right\} \quad \text{— (3)}$$

OR $\dot{x}(t) = A\,x(t)$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

where a, b, c, d are constants. We now develop a method of obtaining a characteristic equation of (3). Let there be a trial solution of (3) of the form

$$\left.\begin{aligned} x_1 &= \alpha e^{\lambda t} \\ x_2 &= \beta e^{\lambda t} \end{aligned}\right\} \quad \text{— (4)}$$

where $e^{\lambda t} \neq 0$ and α, β are constants not both zero, λ is a constant. Therefore from (3) and (4), we obtained

$$\left.\begin{aligned} \lambda \alpha e^{\lambda t} &= a.\alpha e^{\lambda t} + b.\beta e^{\lambda t} \\ \lambda \beta e^{\lambda t} &= c.\alpha e^{\lambda t} + d.\beta e^{\lambda t} \end{aligned}\right\} \quad \text{--- (5)}$$

Since $e^{\lambda t} \neq 0$, then from (5) we get

$$\left.\begin{aligned} (a-\lambda)\alpha + b\beta &= 0 \\ c.\alpha + (d-\lambda)\beta &= 0 \end{aligned}\right\} \quad \text{--- (6)}$$

Now, we would like to find out the value of λ from (6) in view of the fact that ~~sy~~ the system of equations (3) has a sol^n in terms of α and β. For existence of the non-zero sol^{n}s of (6), we must have

$$D = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \text{i.e. } \det(A-\lambda I) = 0$$

$$\text{i.e. } \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad \text{--- (7)}$$

This is known as the <u>characteristic</u> or <u>auxiliary equation</u> of the system of equ^{n}s (3).

Differentiating the first equ^n of (3) with respect to t, we get

$$\frac{d^2x_1}{dt^2} = a\frac{dx_1}{dt} + b\frac{dx_2}{dt}$$

$$= a\frac{dx_1}{dt} + b(cx_1 + dx_2) \quad \text{using 2nd equ}^n \text{ of (3)}$$

$$= a\frac{dx_1}{dt} + bcx_1 + d\left(\frac{dx_1}{dt} - ax_1\right) \quad \text{using 1st equ}^n \text{ of (3)}$$

$$\text{or, } \frac{d^2x_1}{dt^2} - (a+d)\frac{dx_1}{dt} + (ad-bc)x_1 = 0 \quad \text{--- (8)}$$

If $x_1 = \alpha e^{\lambda t}$ be a solⁿ of (8), then the auxiliary equation is

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

which is exactly same as (7).

Hence, to find the solution of the equation (8), we just require the characteristic or auxiliary equation (7) which can easily be obtained by setting $D=0$ i.e $|A-\lambda I| = 0$. Since (7) is a quadratic equation in λ, there are three cases to consider depending on two roots λ_1 and λ_2 which are in turn be real and distinct, real and equal or complex conjugates.

Case I : Distinct Real Roots:

If λ_1 and λ_2 be two real and distinct roots of (7) i.e $|A-\lambda I| = 0$, then

$$x_1 = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad \text{--- (9)}$$

where c_1, c_2 are arbitary constants and from the first equation of (3) we get

$$x_2 = \frac{1}{b}\left(\frac{dx_1}{dt} - a x_1\right)$$

provided $b \neq 0$ and $\frac{dx_1}{dt}$ is to be obtained by differentiating (9) w.r.t. t. Thus, the general solⁿ of (3) i.e $\dot{x}(t) = A x(t)$ is given by

$$\left.\begin{aligned} x_1 &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x_2 &= \frac{1}{b}(\dot{x}_1 - a x_1) \end{aligned}\right\}$$

→ follow lecture note 2

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Lecture Note - 2

Case II : Real and Equal roots

Let $\lambda_1 = \lambda_2 = \lambda$ (say) be two real and equal roots of (7) i.e. $|A-\lambda I|=0$. Then

$$x_1 = (c_1 + c_2 t)e^{\lambda t} \quad \text{———} \quad (10)$$

thus from the first equation of (3) and using (10), we get

$$x_2 = \frac{1}{b}(\dot{x}_1 - a x_1), \text{ provided } b \neq 0.$$

Hence, the general solⁿ of the system (3) is given by

$$\left.\begin{aligned} x_1 &= (c_1 + c_2 t)e^{\lambda t} \\ x_2 &= \frac{1}{b}(\dot{x}_1 - a x_1) \end{aligned}\right\}$$

Case III : Complex Conjugate Roots

Let $\lambda_1 = a_1 + ib_1$ and $\lambda_2 = a_1 - ib_1$ be two complex conjugate roots of (7) ie $|A-\lambda I| = 0$. Then

$$x_1 = e^{a_1 t}(c_1 \cos b_1 t + c_2 \sin b_1 t) \quad \text{———} \quad (11)$$

Thus, from the first equation of (3) and using (11), we get

$$x_2 = \frac{1}{b}(\dot{x}_1 - a x_1), \text{ provided } b \neq 0.$$

Hence, the general solution of the system (3) is given by

$$\left.\begin{aligned} x_1 &= e^{a_1 t}(c_1 \cos b_1 t + c_2 \sin b_1 t) \\ x_2 &= \frac{1}{b}(\dot{x}_1 - a x_1) \end{aligned}\right\}$$

Note: If $b = 0$ and $c \neq 0$, then x_2 is obtained by replacing x_1 in the relation (9), (10) and (11) in three different cases and x_1 is obtained from the second equation of (3) in the form

$$x_1 = \frac{1}{c}(\dot{x}_2 - d x_2)$$

These two relations together constitute the general solution of the given system (3) with c_1, c_2 as two arbitary constants.

Representation of Homogeneous Solutions:

Let $x_1, x_2, \ldots, x_n$ be n linearly independent solutions to the homogeneous system

$$\dot{x}(t) = A x(t) \qquad \text{——— (1)}$$

Then every solution to (1) can be expressed in the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) \quad \text{— (2)}$$

where $c_1, c_2, \ldots, c_n$ are constants.

A set of linearly independent solutions $\{x_1, x_2, \ldots, x_n\}$, or equivalently, whose Wroniskian does not vanish, is called a fundamental solution set for (1). The linear combination in (2), written with arbitrary constants, is called a general solution to (1).

A fundamental matrix for the homogeneous system (1) is

$$X(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]$$

$$= \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{bmatrix}$$

Here $x_i(t) = \begin{pmatrix} x_{1,i}(t) \\ x_{2,i}(t) \\ \vdots \\ x_{n,i}(t) \end{pmatrix}$

Thus we may express the general soln (2) as

$$x(t) = X(t)\,\vec{c},$$

whe $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is an arbitrary constant vector.

Approach to Solving Normal Systems

To determine a general solution to the $n \times n$ homogeneous system $\dot{x}(t) = A\,x(t)$:

(i) Find a fundamental solution set $\{x_1, x_2, \ldots, x_n\}$ that consists of n linearly independent solutions to the homogeneous system.

(ii) Form the linear combination

$$x = X\vec{c} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n ,$$

where $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is any constant vector and

$X = [x_1 \ \ x_2 \ \ x_n]$ is the fundamental matrix,

to obtain a general solution.

Definition: The m vector functions $x_1, x_2, \ldots, x_m$ are linearly dependent on an interval I if there exists constants $c_1, c_2, \ldots, c_m$ not all zero, such that $c_1x_1 + c_2x_2 + \cdots + c_m x_m = \vec{0}$ $\forall$ t in I. If the vectors are not linearly dependent, they are said to be linearly independent on I.

Definition: The Wronskian of n vector functions $x_1(t) = \begin{pmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{n,1} \end{pmatrix}$, $\ldots$, $x_n(t) = \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{n,n} \end{pmatrix}$ is defined to be the function

$$W(x_1, x_2, \ldots, x_n) = \begin{vmatrix} x_{1,1} & x_{1,2} & & x_{1,n} \\ x_{2,1} & x_{2,2} & & x_{2,n} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & & x_{nn} \end{vmatrix}.$$

Remark: vector functions $x_1(t), \ldots, x_n(t)$ are linearly independent on an interval I if there Wronskian is nonzero at any point in I.

Proposition: If vector functions $x_1(t), \ldots, x_n(t)$ are independent solutions to a homogeneous system $\dot{x}(t) = Ax(t)$, then the Wronskian is never zero on I.

Corollary: A set of solns. $x_1, x_2, \ldots, x_n$ to $\dot{x}(t) = Ax(t)$ is linearly independent on I if and only if their Wronskian is never zero on I.

$\longrightarrow$ follow Lecture note - 3

Lecture Note - 3

Some worked out examples

Example 1: Write the given system in normal form (or matrix form)

$$\frac{dx}{dt} = x + y + z$$
$$\frac{dy}{dt} = 2x - y + 3z$$
$$\frac{dz}{dt} = x + 5z$$

Solution: The given system can be expressed as

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t),$$

where $\mathbf{x}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 0 & 5 \end{pmatrix}$

and $\dot{\mathbf{x}}(t) = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix}$

This is the required normal form of the system.

Example 2: Express the following 3rd order linear differential eqn in normal form.

$$\frac{d^3y}{dt^3} - \frac{dy}{dt} + y = \cos t$$

<u>solution:</u> The given diff. equ^n can be written as a first-order system in normal form using the substitution

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \text{ and } x_3(t) = y''(t)$$

Thus, $\dot{x}_1 = y' = x_2$, $\dot{x}_2 = y'' = x_3$ and $\dot{x}_3 = y''' = -y + y' + \cos t$ ~~and [illegible]~~.

$$= -x_1 + x_2 + \cos t$$

The given equ^n is equivalent to

$$\dot{x}(t) = A\,x(t) + f(t) \quad \text{—— (1)}$$

where $x(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $f(t) = \begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix}$

and $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

Therefore (1) is the required normal form. <u>Ans</u>.

Check that $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix}$

$$= \begin{pmatrix} x_2 \\ x_3 \\ -x_1 + x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cos t \end{pmatrix}$$

$$\Rightarrow \quad \dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -x_1 + x_2 + \cos t$$

Example 3: Determine whether the given vector functions are linearly independent or linearly dependent on the interval $(-\infty, \infty)$.

$$\begin{pmatrix} te^{-t} \\ e^{-t} \end{pmatrix}, \quad \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

Ans: let $x_1 = \begin{pmatrix} te^{-t} \\ e^{-t} \end{pmatrix}$ and $x_2 = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$

The Wronskian of x_1 & x_2 is

$$W(x_1, x_2) = \begin{vmatrix} te^{-t} & e^{-t} \\ e^{-t} & e^{-t} \end{vmatrix} = (t-1)\, e^{-2t}$$

At point $t = 1$ in the interval $(-\infty, \infty)$, the Wronskian $W(x_1, x_2)$ become zero. Hence, the given vector functions are linearly dependent.

Example 4: The vector functions $x_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}$, $x_2 = \begin{pmatrix} \sin t \\ \cos t \\ -\sin t \end{pmatrix}$ and $x_3 = \begin{pmatrix} -\cos t \\ \sin t \\ \cos t \end{pmatrix}$ are solutions to a system $\dot{x}(t) = Ax(t)$. Determine whether they form a fundamental solution set. If they do, find a fundamental matrix for the system and give a general solution.

Solution: The Wronskian of the vector functions x_1, x_2, x_3 is

$$W(x_1, x_2, x_3) = \begin{vmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{vmatrix}$$

$$= \begin{vmatrix} e^t & \sin t & -\cos t \\ 0 & \cos t - \sin t & \sin t + \cos t \\ 0 & -2\sin t & 2\cos t \end{vmatrix} \quad \begin{matrix} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{matrix}$$

$$= e^t \begin{vmatrix} \cos t - \sin t & \sin t + \cos t \\ -2\sin t & 2\cos t \end{vmatrix}$$

$$= e^t \left[2\cos^2 t - \cancel{2\sin t \cos t} + 2\sin^2 t + \cancel{2\sin t \cos t} \right]$$

$$= 2e^t \neq 0$$

So the vector functions x_1, x_2, x_3 are linearly independent.

Thus $\{x_1, x_2, x_3\}$ ~~is~~ forms a fundamental solution set of the system $\dot{x}(t) = A x(t)$.

The fundamental matrix for the system is

$$[x_1 \; x_2 \; x_3] = \begin{pmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{pmatrix}$$

The general solution of the system is

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t),$$

where c_1, c_2, c_3 are arbitrary constants.

→ follow Lecture Note - 4

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Lecture Note - 4

Example 5 : Write down the normal linear system for the following differential equation of order 3:
(WBSU-2019)

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} - 4x = te^{-t}$$

Solution: Let $x_1 = x$, $x_2 = \dot{x} = \frac{dx}{dt}$ and $x_3 = \ddot{x} = \frac{d^2x}{dt^2}$

$$\therefore \dot{x}_1 = \frac{dx}{dt} = x_2$$

$$\dot{x}_2 = \ddot{x} = x_3$$

$$\dot{x}_3 = \dddot{x} = 4x - 3\ddot{x} + te^{-t} \quad \text{[from given diff. equ}^n\text{]}$$

$$= 4x_1 - 3x_3 + te^{-t}$$

Thus the given diff. equ^n of order 3 can be expressed as following normal form

$$\dot{x}(t) = Ax(t) + f(t),$$

where $x(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -3 \end{pmatrix}$

and $f(t) = \begin{pmatrix} 0 \\ 0 \\ te^{-t} \end{pmatrix}$.

Example 6 : Find a fundamental matrix for the linear
(WBSU-2019) system $\dot{x}(t) = Ax(t)$, where

$$A = \begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Solution : Assuming a solⁿ of given system of the form

$$\bar{x}(t) = \bar{\alpha} e^{\lambda t}, \quad \bar{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

that is, $x_1(t) = \alpha_1 e^{\lambda t}$, $x_2(t) = \alpha_2 e^{\lambda t}$,

We know that λ must be a solⁿ of the characteristic equⁿ of the coefficient matrix $A = \begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$.

The characteristic equⁿ is

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 3 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda)^2 - 9 = 0$$

$$\Rightarrow (2+\lambda)^2 = 9 \Rightarrow 2 + \lambda = \pm 3$$

$$\Rightarrow \lambda = -5, 1$$

The characteristic vector $\bar{\alpha}$ corresponding to $\lambda = -5$ is obtain from the equⁿ

$$A\bar{\alpha} = \lambda \bar{\alpha}$$

$$\Rightarrow (A - \lambda)\bar{\alpha} = 0$$

$$\Rightarrow \begin{pmatrix} -2-\lambda & 3 \\ 3 & -2-\lambda \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ since } \lambda = -5$$

$$\Rightarrow \alpha_1 + \alpha_2 = 0$$

A simple non trivial sol^n ~~of~~ this is $\alpha_1 = 1, \alpha_2 = -1$

Thus a characteristic vector corresponding to $\lambda = -5$ is

$$\bar{\alpha} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $$x(t) = \bar{\alpha} e^{\lambda t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} = \begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix},$$ is a

sol^n of given system.

For $\lambda = 1$, characteristic vector $\bar{\alpha}$ is obtain by

$$\begin{pmatrix} -2-\lambda & 3 \\ 3 & -2-\lambda \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\Rightarrow \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ since } \lambda = 1$$

$$\Rightarrow \alpha_1 = \alpha_2$$

A simple non trivial sol^n of this is $\alpha_1 = 1, \alpha_2 = 1$.

Thus a characteristic vector corresponding to $\lambda = 1$ is

$$\bar{\alpha} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So $$x(t) = \bar{\alpha} e^{\lambda t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$$ is a sol^n

of given system.

The solutions $\begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix}$ and $\begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$ are linearly

independent.

So the fundamental matrix of the given system

is $$\begin{pmatrix} e^{-5t} & e^{t} \\ -e^{-5t} & e^{t} \end{pmatrix}$$ Ans.

and a general sol^n is $x(t) = c_1 \begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix} + c_2 \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$.

Alternative method :-

The given system of equⁿs are

$$\left.\begin{aligned}\frac{dx_1}{dt} &= -2x_1 + 3x_2 \\ \frac{dx_2}{dt} &= 3x_1 - 2x_2\end{aligned}\right\} \quad (1)$$

let $x_1 = \alpha_1 e^{\lambda t}$ & $x_2 = \alpha_2 e^{\lambda t}$, be the non-trivial solution of (1), where α_1, α_2 are constants not both zero and λ is a constant.

The auxiliary equⁿ or characteristic equⁿ of (1) is

$$\begin{vmatrix} -2-\lambda & 3 \\ 3 & -2-\lambda \end{vmatrix} = 0 \Rightarrow 2+\lambda = \pm 3$$

$$\Rightarrow \lambda = -5, 1$$

(Real and distinct roots)

Therefore $x_1 = c_1 e^{-5t} + c_2 e^{t}$, where c_1, c_2 are arbitrary ~~consts~~ constants.

Now from first equation of (1), we get

$$3x_2 = 2x_1 + \frac{dx_1}{dt}$$

$$= 2(c_1 e^{-5t} + c_2 e^{t}) + (-5c_1 e^{-5t} + c_2 e^{t})$$

$$= -3c_1 e^{-5t} + 3c_2 e^{t}$$

$$\Rightarrow x_2 = c_1(-e^{-5t}) + c_2 e^{t}$$

Thus the general solⁿ of the given system (1) is

$$x_1 = c_1 e^{-5t} + c_2 e^{t}$$

and $x_2 = c_1(-e^{-5t}) + c_2 e^{t}$

Hence the fundamental matrix of the given system is

$$\begin{pmatrix} e^{-5t} & e^{t} \\ -e^{-5t} & e^{t} \end{pmatrix}$$ Ans.

Example 7: Solve $\frac{dx}{dt} = -2x+7y$, $\frac{dy}{dt} = 3x+2y$; given that $x=9$, $y=-1$ when $t=0$.

Solution: The given system of equations are

$$\left.\begin{aligned} \frac{dx}{dt} &= -2x+7y \\ \frac{dy}{dt} &= 3x+2y \end{aligned}\right\} \quad \text{--- (1)}$$

Let $x = \alpha_1 e^{\lambda t}$, $y = \alpha_2 e^{\lambda t}$ be the non-trivial solution of the given system (1), where α_1, α_2 are constants not both zero and λ is a constant.

The auxiliary eqn of the given system is

$$\begin{vmatrix} -2-\lambda & 7 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5$$

(real and distinct roots

Therefore, $x = c_1 e^{-5t} + c_2 e^{5t}$, where c_1, c_2 are arbitrary constants.

Now, from the first equation of (1), we get

$$\begin{aligned} 7y &= 2x + \frac{dx}{dt} \\ &= 2(c_1 e^{-5t} + c_2 e^{5t}) + (-5c_1 e^{-5t} + 5c_2 e^{5t}) \\ &= -3c_1 e^{-5t} + 7c_2 e^{5t} \end{aligned}$$

$$\Rightarrow y = c_1\left(-\frac{3}{7}e^{-5t}\right) + c_2 e^{5t}$$

Thus, the general solution of the given system (1) is

$$\left.\begin{aligned} x &= c_1 e^{-5t} + c_2 e^{5t} \\ y &= c_1\left(-\frac{3}{7}e^{-5t}\right) + c_2 e^{5t} \end{aligned}\right\} \text{--- (2)}$$

The fundamental matrix of the given system is

$$\begin{pmatrix} e^{-5t} & e^{5t} \\ -\frac{3}{7}e^{-5t} & e^{5t} \end{pmatrix}$$

Using given initial conditions, from (2), we get

$$9 = c_1 + c_2 \quad \text{and} \quad -1 = \cancel{\;} -\tfrac{3}{7}c_1 + c_2$$

Solving these relations, we get

$$c_1 = 7 \quad \text{and} \quad c_2 = 2$$

Hence, the solution of the given problem is

$$x = 7e^{-5t} + 2e^{5t} \quad \text{and} \quad y = -3e^{-5t} + 2e^{5t}$$

Example 8 : Solve : $\frac{dx}{dt} = 4x+y$, $\frac{dy}{dt} = -8x+8y$

Solution : The given system is

$$\left.\begin{aligned} \frac{dx}{dt} &= 4x+y \\ \frac{dy}{dt} &= -8x+8y \end{aligned}\right\} \quad \text{—— (1)}$$

Let $x = \alpha_1 e^{\lambda t}$, $y = \alpha_2 e^{\lambda t}$ be the non-trivial solution of the given system (1), where α_1, α_2 are constants not both zero and λ is a constant.

The auxiliary eqn for the given system (1) is

$$\begin{vmatrix} 4-\lambda & 1 \\ -8 & 8-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 12\lambda + 40 = 0$$

$$\Rightarrow \lambda = 6 \pm 2i$$

Therefore $x = e^{6t}(c_1 \cos 2t + c_2 \sin 2t)$; where c_1, c_2 are two arbitrary constants. —— (2)

Now, from first equation of (1) we get

$$y = -4x + \frac{dx}{dt}$$

$$= -4e^{6t}(c_1\cos 2t + c_2 \sin 2t) + 2e^{6t}\{(3c_1+c_2)\cos 2t + (3c_2 - c_1)\sin 2t\}$$

$$= 2e^{6t}\{(c_1+c_2)\cos 2t + (c_2-c_1)\sin 2t\} \quad \text{—— (3)}$$

Thus, (2) and (3) together give the required solution of the system (1). Ans.

Example 9 : Solve $\frac{dx}{dt} = 5x + 4y$, $\frac{dy}{dt} = -x + y$.

Solution : Given equations are

$$\left.\begin{aligned} \frac{dx}{dt} &= 5x + 4y \\ \frac{dy}{dt} &= -x + y \end{aligned}\right\} \quad \text{--- (1)}$$

Let $x = \alpha_1 e^{\lambda t}$, $y = \alpha_2 e^{\lambda t}$ be the non-trivial solution of the given system (1), where α_1, α_2 are constants not both zero and λ is a constant.

The auxiliary equation of the given system (1) is

$$\begin{vmatrix} 5-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow \lambda = 3, 3 \quad \text{(real and equal roots)}$$

Thus, $y = (c_1 + t c_2) e^{3t}$ where c_1, c_2 are arbitrary constants. (2)

Now, using 2nd equation of (1) we get

$$\begin{aligned} x &= y - \frac{dy}{dt} \\ &= (c_1 + t c_2) e^{3t} - 2(c_1 + t c_2) e^{3t} - c_2 e^{3t} \\ &= -2c_1 e^{3t} - (2t+1) c_2 e^{3t} \quad \text{--- (3)} \end{aligned}$$

Hence, the eqn's (2) and (3) give the required solution of the given system (1).

~~Example 10/12~~

Exercises - 1

Solve the following simultaneous differential equations;

1. $\frac{dx}{dt} = x - 2y$, $\frac{dy}{dt} = 5x + 3y$
2. $\frac{dx}{dt} = -3x + 4y$, $\frac{dy}{dt} = -2x + 3y$
3. $\frac{dx}{dt} = -4x - y$, $\frac{dy}{dt} = x - 2y$

4. $\frac{dx}{dt} = x + 3y, \quad \frac{dy}{dt} = 3x + y$

5. $\frac{dx}{dt} = 3x - y, \quad \frac{dy}{dt} = 4x - y$

6. $\frac{dx}{dt} = 3x + 2y, \quad \frac{dy}{dt} = -3x - 3y$

7. $\frac{dx}{dt} + x - y = 0, \quad \frac{dy}{dt} + 2x + 5y = 0$

8. $\frac{dx}{dt} = x - 2y, \quad \frac{dy}{dt} = 4x + 5y$

9. $\frac{dx}{dt} = 3x + 2y, \quad \frac{dy}{dt} = 5x + 3y$

Find the particular solution of the linear system that satisfie the stated initial conditions.

10. $\frac{dx}{dt} = -2x + 7y, \quad \frac{dy}{dt} = 3x + 2y, \quad x(0) = 9, \; y(0) = -1$

11. $\frac{dx}{dt} = -2x + y, \quad \frac{dy}{dt} = 7x + 4y, \quad x(0) = 6, \; y(0) = 2$

12. $\frac{dx}{dt} = 2x - 8y, \quad \frac{dy}{dt} = x + 6y, \quad x(0) = 4, \; y(0) = 1$

13. $\frac{dx}{dt} = 3x + 5y, \quad \frac{dy}{dt} = -2x + 5y, \quad x(0) = 5, \; y(0) = -1.$

14. $\frac{dx}{dt} = 6x - 4y, \quad \frac{dy}{dt} = x + 2y, \quad x(0) = 2, \; y(0) = 3$

15. $\frac{dx}{dt} = 7x - y, \quad \frac{dy}{dt} = 4x + 3y, \quad x(0) = 1, \; y(0) = 3$

$\longrightarrow$ follow lecture note 5