LECTURE NOTE-3

Solution of Non-Linear PARTIAL DIFFERENTIAL EQUATIONS

CHARPIT'S METHOD:

General method of solving partial differential equations of order one but of any degree in two independent variables x and y

Let us consider the partial differential equation of first order and non-linear in p and q as

$$f(x,y,z,p,q)=0 \quad \text{———} (1)$$

We know that $dz = p\,dx + q\,dy$ ——— (2)

The next step is to find another relation

$$F(x,y,z,p,q)=0 \quad \text{———} (3)$$

such that when the values of p and q obtained by solving (1) and (3) are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (3), differentiating partially (1) and (3) with respect to x and y, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x}=0 \quad \cdots\cdots (4)$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\cdot\frac{\partial q}{\partial x}=0 \quad \cdots\cdots (5)$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0 \quad \cdots\cdots (6)$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0 \quad \cdots\cdots (7)$$

where $\frac{\partial z}{\partial x}=p$ and $\frac{\partial z}{\partial y}=q$

Now, eliminating $\frac{\partial p}{\partial x}$ from (4) and (5) we get

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p}-\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p}=0$$

or, $$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p+\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x}=0 \quad \cdots\cdots (8)$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ from (6) and (7) we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q+\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}=0 \quad \text{.......................(9)}$$

Since $\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial p}{\partial y}$, the last term in (8) is same as that in (9), except for a minus sign and hence they cancel on adding (8) and (9)

Therefore, adding (8) and (9) and rearranging the terms, we get

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}+\left(\frac{-\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(\frac{-\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0 \quad \text{..............(10)}$$

This is a linear equation of the first order to obtain the desired function F.

The integral of (10) is obtained by solving the auxliary equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0} \quad \text{...................(11)}$$

Since any of the integrals of (11) will satisfy (10), an integral of (11) which involves p or q (or both) will serve along with the given equation to find p and q. For practice, we shall select the simplest integral. Using the standard notations :

$f_x=\frac{\partial f}{\partial x}$, $f_y=\frac{\partial f}{\partial y}$, $f_z=\frac{\partial f}{\partial z}$, $f_p=\frac{\partial f}{\partial p}$, $f_q=\frac{\partial f}{\partial q}$; Charpit's auxliary equation (11) may be re-written as

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dF}{0} \quad \text{.............(11)'}$$

Working Rule for Charpit's Method

STEP 1 Transfer all terms of the given equation to L.H.S. and denote the entire expression by f.

STEP 2 Write down the Charpit's auxliary equations 11 or (11)′.

STEP 3 Using the value of f in Step 1, write down the values of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, ... i.e., f_x, f_y, ... etc occuring in **Step 2** and put these in Charpit's equation 11 or (11)′.

STEP 4 After simplifying the **Step 3**, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

STEP 5 In the simplest relation of **Step 4** is solved along with the given equation to determine p and q. Put these values of p and q in $dz = pdx + qdy$ which on integration gives the complete integral of the given equation. The singular and the general integrals may be obtained in usual manner.

Note : Sometimes Charpit's equations gives $p = a$ and $q = p$, where a and b are constants. In such cases, putting $p = a$ and $q = b$ in the given equations will give the required complete integral.

EXAMPLES WORKED OUT

1 ■▶ ***Apply Charpit's method to find a complete integral of $z = pq$.***

Let $f(x, y, z, p, q) \equiv z - pq = 0$ .... (1)

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{..................... (2)}$$

From (1), $f_x = 0, f_y = 0,\ f_z = 1,\ f_p = -q, f_q = -p$ .... (3)

Using (3), (2) reduces to

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{2pq} = \frac{dx}{q} = \frac{dy}{p} \quad \text{..................... (4)}$$

Taking the first two fractions of (4) we get

$$\frac{dp}{p} = \frac{dq}{q}. \text{ Integrating, } \log p = \log q + \log a = \log aq$$

or, $\quad p = aq$, where a is a constant

$\therefore$ From (1), $z = aq^2$ or, $q = \sqrt{\frac{z}{a}}$.

$\therefore\ p = a \cdot q = a \cdot \sqrt{\frac{z}{a}} = \sqrt{az}$

$\therefore\ dz = pdx + qdy = \sqrt{az}dx + \sqrt{\frac{z}{a}}dy$

or, $\quad \frac{dz}{\sqrt{z}} = \sqrt{a}dx + \frac{1}{\sqrt{a}}dy$

Integrating, $2\sqrt{z} = \sqrt{a}x + \frac{1}{\sqrt{a}}y + b$, where a and b are constants.

2 ■▶ ***Find a complete integral of $px + qy = pq$, by Charpit's method.***

Here the given equation is

$$f(x, y, z, p, q) \equiv px + qy - pq = 0 \quad \text{.............. (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

From (1), we get $f_x = p, f_y = q,\ f_z = 0, f_p = x - q,\ f_q = y - p$

$\therefore$ From above we get

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} \quad \text{..................... (2)}$$

Taking the first two fractions of (2) we get

$$\frac{dp}{p}=\frac{dq}{q}.\text{ Integrating } \log p = \log q + \log a$$

or, $p = aq$, where a is a constant .... (3)

Putting the value of p in (1) we get

$$aqx + qy - aq^2 = 0 \text{ or, } aq = ax + y \text{ or, } q = \frac{ax+y}{a} \text{ .... (4)}$$

$\therefore$ From (3) $p = ax + y$ .... (5)

Putting the value of p and q in $dz = pdx + qdy$ we get

$$dz = (ax+y)dx + \frac{(ax+y)}{a}dy$$

or, $adz = (ax + y)(adx + dy) = (ax + y)\, d\,(ax + y)$

Integrating, $az = \frac{(ax+y)^2}{2} + b$, which is a complete integral, where a, b are constants.

3 ■▶ ***Find the complete integral of $zpq = p + q$, by using Charpit's Method.***

Let $f(x, y, z, p, q) \equiv zpq - p - q = 0$ .... (1)

Here Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \text{ .... (2)}$$

From (1), $f_x = 0, f_y = 0, f_z = pq, f_p = zq - 1, f_q = zp - 1$

Putting these values in (2), we get

$$\frac{dp}{p^2q} = \frac{dq}{pq^2} \text{ from first and second ratio}$$

or, $\frac{dp}{p} = \frac{dq}{q}$. Integrating, $\log p = \log q + \log a = \log aq$

or, $p = aq$, where a is an arbitrary constant

$\therefore$ From (1) we get $zpq = p + q = aq + q = (a + 1)q$ or, $p = \frac{a+1}{z}$

$\therefore\ q = \frac{p}{a} = \frac{a+1}{az}$.

$\therefore\ dz = pdx + qdy = \frac{1+a}{z}dx + \frac{a+1}{az}dy$

or, $2zdz = 2(1+a)dx + \frac{2(1+a)}{a}dy = 2(1+a)\left[dx + \frac{1}{a}dy\right]$

Integrating, $z^2 = 2(1+a)\left[x + \frac{y}{a}\right] + b$, where a and b are constants, which is the complete integral.

4 ■▶ ***Find a complete integral of $z^2 = pqxy$, by using Charpit's method.***

The given equation is

$f(x, y, z, p, q) \equiv z^2 - pqxy = 0$ .... (1)

Here Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{..............} (2)$$

From (1), $f_x = -pqy$, $f_y = -pqx$, $f_z = 2z$, $f_p = -qxy$, $f_q = -pxy$.

$\therefore$ From (2) we get, $\dfrac{dp}{-pqy + 2pz} = \dfrac{dq}{-pqx + 2qz} = \dfrac{dz}{2pqxy} = \dfrac{dx}{qxy} = \dfrac{dy}{pxy}$ .... (3)

Each part of (3)

$$= \frac{xdp + pdx}{x(-pqy + 2pz) + pqxy} = \frac{ydq + qdy}{y(-pqx + 2qz) + pqxy}$$

or, $\dfrac{xdp + pdx}{2pxz} = \dfrac{ydq + qdy}{2qyz}$ or, $\dfrac{d(xp)}{xp} = \dfrac{d(yq)}{yq}$.

Integrating, log (px) = log (qy) +log a^2, where a is an integrating constant

or, $xp = a^2qy$ .... (4)

$\therefore$ From (1) and (4) we get

$$z^2 = (px)(qy) = a^2(qy)^2$$

or, $(qy)^2 = \dfrac{z^2}{a^2}$ or, $qy = \dfrac{z}{a}$ or, $q = \dfrac{z}{ay}$

$\therefore$ $p = \dfrac{a^2qy}{x} = \dfrac{a^2}{x} \cdot \dfrac{z}{a} = \dfrac{az}{x}$.

Putting the value of p and q in $dz = pdz + qdy$

we get $dz = \dfrac{az}{x} dx + \dfrac{z}{ay} dy$

or, $\dfrac{dz}{z} = a\dfrac{dx}{x} + \dfrac{1}{a} \cdot \dfrac{dy}{y}$

or, $\log z = a \log x + \dfrac{1}{a} \log y + \log b$, where b is an arbitrary constant

or, $z = x^a y^{1/a} b$, which is the required complete integral.

7 ▶ ***Find a complete integral of $q = 3p^2$, by Charpit's method.***

The given equation is

$$f(x, y, z, p, q) \equiv 3p^2 - q = 0 \quad \text{..............} (1)$$

Here $f_x = 0$, $f_y = 0$, $f_z = 0$, $f_p = 6p$, $f_q = -1$

Charpit's auxliary equations are

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-6p^2 + q} = \frac{dx}{-6p} = \frac{dy}{1}$$

Taking the first fraction, we get $p = a$.

Putting the value of p in (1) we get $q = 3a^2$ .... (2)

Now, putting the value of p and q in $dz = pdx + qdy$ we get $dz = adx + 3a^2dy$

Integrating, $z = ax + 3a^2y + b$, which is a complete integral, where a and b are arbitrary constants.

8 ■▶ ***Using Charpit's method, find the complete integral of $(p^2 + q^2)\, y = qz$.***

The given equation is

$$f(x, y, z, p, q) \equiv (p^2 + q^2)\, y - qz = 0 \text{ .... (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \text{ .... (2)}$$

Here $f_x = 0, f_y = p^2 + q^2, f_z = -q, f_p = 2py, f_q = 2qy - z$

$\therefore$ From (2) we get

$$\frac{dp}{-pq} = \frac{dq}{p^2 + q^2 - q^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{-2qy + z}$$

or, $$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2(p^2 + q^2)y + qz} = \frac{dx}{-2py} = \frac{dy}{z - 2qy}$$

Taking the first and second parts of the equation

$$pdp + qdq = 0 \quad \text{or, } 2pdp + 2qdq = 0$$

Integrating, $p^2 + q^2 = a^2$ .... (3), where a^2 is an arbitrary constant.

Using (1) and (3) we get

$$a^2y - qz = 0 \quad \text{or, } q = \frac{a^2y}{z}.$$

Putting the value of q in (3) we get

$$p^2 = a^2 - \frac{a^4y^2}{z^2} = \frac{a^2}{z^2}(z^2 - a^2y^2) \quad \text{or, } p = \frac{a}{z}\sqrt{z^2 - a^2y^2}$$

Putting these values of p and q in

$dz = pdx + qdy$ we get

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\, dx + \frac{a^2y}{z}\, dy$$

or, $$\frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y^2}} = adx \quad \text{or, } d\left\{\sqrt{z^2 - a^2y^2}\right\} = adx.$$

Integrating, $\sqrt{z^2 - a^2y^2} = ax + b$, b being an integrating constant.

or, $(z^2 - a^2y^2) = (ax + b)^2$, which is the required complete integral.

9 ■▶ ***Find a complete integral of $p^2x + q^2y = z$, by Charpit's method.***

The given equation is

$$f(x, y, z, p, q) \equiv p^2x + q^2y - z = 0 \text{ .... (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{.............. (2)}$$

Here $f_x = p^2, f_y = q^2, f_z = -1, f_p = 2px, f_q = 2qy$

$\therefore$ From (2) we get

$$\frac{dp}{p^2 - p} = \frac{dq}{q^2 - q} = \frac{dz}{-2(p^2x + q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy} \quad \text{.............. (3)}$$

Each fraction of (3) is equal to

$$\frac{2pxdp + p^2dx}{2px(p^2 - p) + p^2(-2px)} = \frac{2qydq + q^2dy}{2qy(q^2 - q) + q^2(-2qy)}$$

or, $$\frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2q^2y}$$

Integrating, $\log(p^2x) = \log(q^2y) + \log a$, where a is a constant of integration

or, $p^2x = aq^2y$ .... (3)

From (3) $p^2 = \frac{aq^2y}{x}$

Putting the value of p^2 in (1) we get

$$\frac{aq^2y}{x} \cdot x + q^2y - z = 0$$

or, $q^2y(1 + a) = z$ or, $q = \sqrt{\frac{z}{(1+a)y}}$ $\therefore p = \sqrt{\frac{za}{(1+a)x}}$

with these values of p and q, we get from

$dz = pdx + qdy$ as

$$dz = \sqrt{\frac{za}{(1+a)x}}dx + \sqrt{\frac{z}{(1+a)y}}dy$$

or, $$\sqrt{1+a}\frac{dz}{\sqrt{z}} = \sqrt{a}\frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}$$

Integrating, $2\sqrt{1+a} \cdot \sqrt{z} = 2\sqrt{a} \cdot \sqrt{x} + 2\sqrt{y} + 2b$

or, $\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$, b being the constant of integration, which is the required complete integral.

10 ■▶ ***Using Charpit's method, find a complete integral of $z^2(p^2z^2 + q^2) = 1$.***

The given equation is

$$f(x, y, z, p, q) \equiv p^2z^4 + z^2q^2 - 1 = 0 \quad \text{.............. (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{.............. (2)}$$

Here $f_x = 0, f_y = 0, f_z = 4p^2z^3 + 2zq^2, f_p = 2pz^4, f_q = 2qz^2$

$\therefore$ From (1) we get

$$\frac{dp}{p(4p^2z^3+2zq^2)}=\frac{dq}{q(4p^2z^3+2zq^2)}=\frac{dz}{-2p^2z^4-2q^2z^2}=\frac{dx}{-2pz^4}=\frac{dy}{-2qz^2}$$

Taking the first two parts, we get

$\frac{dp}{p}=\frac{dq}{q}$. Integrating $p = aq$ .... (2), a is a constant of integration

Putting the value of p in (1) we get

$$a^2q^2z^4 + z^2q^2 - 1 = 0 \quad \text{or, } z^2q^2 (a^2z^2 + 1) = 1$$

$$\text{or,} \quad q=\frac{1}{z\sqrt{a^2z^2+1}} \quad \therefore p=\frac{a}{z\sqrt{a^2z^2+1}}$$

Putting the value of p and q in $dz = pdx + qdy$ we get

$$dz=\frac{adx}{z\sqrt{a^2z^2+1}}+\frac{dy}{z\sqrt{a^2z^2+1}}$$

$$\text{or,} \quad dz=\frac{adx+dy}{z\sqrt{a^2z^2+1}} \quad \text{or,} \quad adx+dy=z\sqrt{a^2z^2+1}\,dz$$

Integrating, we get

$$ax+y+b=\frac{1}{3a^2}(a^2z^2+1)^{\frac{3}{2}},$$ where b is the constant of integration

$$\text{or,} \quad 3a^2(ax+y+b)=(a^2z^2+1)^{\frac{3}{2}}$$

$$\text{or,} \quad 9a^4 (ax + y + b)^2 = (a^2z^2 + 1)^3,$$ which is the required complete integral.

11 ■▶ ***Find a complete integral of $(p + q)(px + qy) = 1$, by Charpit's method.***

The given equation is

$$f(x, y, z, p, q) \equiv (p + q)(px + qy) - 1 \text{ .... (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q} \text{ .... (2)}$$

Here $f_x = p(p + q), f_y = q(p + q), f_z = 0, f_p = 2px + qy + qx, f_q = px + 2qy + py$

$\therefore$ From (2) we get

$$\frac{dp}{p(p+q)}=\frac{dq}{q(p+q)}=\frac{dz}{-2(p+q)(px+qy)}=\frac{dx}{-(2px+qx+qy)}=\frac{dy}{-(2qy+px+py)} \text{ .... (3)}$$

Taking the first two ratios of (3) we get

$\frac{dp}{p}=\frac{dq}{q}$. Integrating, $\log p = \log q + \log a = \log aq$

or, $p = aq$ .... (3), where a is a constant of integration

Putting $p = aq$ in (1) we get

$$q(1 + a)(aqx + qy) = 1 \text{ or, } q^2 (1 + a)(ax + y) = 1$$

$$\text{or,} \quad q=\frac{1}{\sqrt{(1+a)(ax+y)}} \quad \therefore p=aq=\frac{a}{\sqrt{(1+a)(ax+y)}}$$

Putting the value of p and q in $dz = pdx + qdy$ we get

$$dz = \frac{a}{\sqrt{(1+a)(ax+y)}}dx + \frac{1}{\sqrt{(1+a)(ax+y)}}dy$$

or, $$dz = \frac{adx+dy}{\sqrt{(1+a)(ax+y)}} = \frac{adx+dy}{\sqrt{(1+a)}\sqrt{(ax+y)}} = \frac{d(ax+y)}{\sqrt{1+a}\sqrt{ax+y}}$$

or, $$\sqrt{1+a}dz = \frac{d(ax+y)}{\sqrt{ax+y}}$$

Integrating, $\sqrt{1+a}\ z = 2a\sqrt{ax+y} + b$, where b is the constant of integration, which is the required complete integral.

12 ■▶ ***Find a complete integral of*** $xp + 3yq = 2(z - x^2q^2)$ **by charpit's method.**

The given equation is

$$f(x, y, z, p, q) \equiv xp + 3yq - 2z + 2x^2q^2 = 0 \text{ .... (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \text{ .... (2)}$$

Here $f_x = p + 4xq^2, f_y = 3q, f_z = -2, f_p = x, f_q = 3y + 4qx^2$

∴ From (2) we get

$$\frac{dp}{p+4xq^2-2p} = \frac{dq}{3q-2q} = \frac{dz}{-px-q(3y+4qx^2)} = \frac{dx}{-x} = \frac{dy}{-(3y+4qx^2)}$$

or, $$\frac{dp}{-p+4xq^2} = \frac{dq}{q} = \frac{dz}{-px-q(3y+4qx^2)} = \frac{dx}{-x} = \frac{dy}{-(3y+4qx^2)} \text{ .... (3)}$$

From 2nd and 4th ratio of (3) we get

$$\frac{dq}{q} = \frac{-dx}{x} \text{ or, } \frac{dq}{q} + \frac{dx}{x} = 0$$

Integrating, $\log qx = \log a$ or, $q = \frac{a}{x}$, where 'a' is an integrating constant.

Putting the value of q in (1) we get

$$xp + 3y\left(\frac{a}{x}\right) - 2z + 2x^2\left(\frac{a^2}{x^2}\right) = 0$$

or, $$px = 2z - 2a^2 - \frac{3ay}{x} \text{ or, } p = \frac{2(z-a^2)}{x} - \frac{3ay}{x^2}$$

Putting the value of p and q in $dz = pdx + qdy$ we get

$$dz = \left\{\frac{2(z-a^2)}{x} - \frac{3ay}{x^2}\right\}dx + \frac{a}{x}dy$$

or, $x^2dz = 2x(z - a^2)\ dx - 3aydx + axdy$

or, $x^2dz - 2x(z - a^2)\ dx = -\ 3aydx + axdy$

or, $$\frac{x^2dz - 2x(z-a^2)dx}{x^4} = \frac{-3aydx}{x^4} + \frac{ady}{x^3} \text{ or, } d\left(\frac{z-a^2}{x^2}\right) = d\left(\frac{ay}{x^3}\right)$$

Integrating, $\frac{z-a^2}{x^2} = \frac{ay}{x^3} + b$, b being an integrating constant.

or, $z = \frac{ay}{x} + bx^2 + a^2 = a\left(a + \frac{y}{x}\right) + bx^2$, which is the required complete integral.

Putting these values of p and q in $dz = pdx + qdy$ we get

$$dz = \frac{(ax+y) \pm \sqrt{(ax+y)^2 - (a^2+1)}}{(a^2+1)}(adx + dy) \quad \text{.......... (4)}$$

Let us put $v = ax + y \quad \therefore \ adx + dy = dv$

$\therefore$ From (4) we get

$$(a^2+1)dz = \left\{v \pm \sqrt{v^2 - (a^2+1)}\right\} dv$$

Integrating, $(a^2+1)z = \frac{v^2}{2} \pm \left[\frac{v}{2} \cdot \sqrt{v^2 - (a^2+1)} - \frac{1}{2}(a^2+1)\log(v + \sqrt{v^2 - (a^2+1)}\right] + b$

which is the required complete integral and a, b are arbitrary constants and $v = ax + y$.

16 ■▶ ***Find a complete integral of $p^2 + q^2 - 2px - 2qy + 2xy = 0$, by Charpit's method.***

The given equation is

$$f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0 \quad \text{................ (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{.............. (2)}$$

Here $f_x = -2p + 2y, f_y = -2q + 2x, f_z = 0, f_p = 2p - 2x, f_q = 2q - 2y$.

$\therefore$ From (2) we get

$$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dz}{-p(2p-2x) - q(2q-2y)} = \frac{dp}{2x-2p} = \frac{dq}{2y-2q}$$

Each ratio is equal to

$$\frac{dp+dq}{2(x+y-p-q)} = \frac{dx+dy}{2(x+y-p-q)}$$

which gives $dp + dq = dx + dy$ or, $dp - dx + dq - dy = 0$

Integrating, $(p - x) + (q - y) = a$ .... (3), where a is the constant of integration.

Also, from (1) we get

$$(p-x)^2 + (q-y)^2 = (x-y)^2 \quad \text{................ (4)}$$

From (2) $q - y = a - (p - x)$.

Putting the value of $q - y$ in (4) we get

$$(p-x)^2 + [a - (p-x)]^2 = (x-y)^2$$

or, $2(p-x)^2 - 2a(p-x) + \{a^2 - (x-y)^2\} = 0$

$$\therefore \quad p - x = \frac{2a \pm \sqrt{4a^2 - 4 \cdot 2\{a^2 - (x-y)^2\}}}{4}$$

or, $p = x + \frac{1}{2}\left[a \pm \sqrt{\{2(x-y)^2 - a^2\}}\right]$

$\therefore$ From (3) $q = a + y - p + x = y + \frac{1}{2}\left[a \mp \sqrt{\{2(x-y)^2 - a^2\}}\right]$

Putting the value of p and q in $dz = pdx + qdy$ we get

$$dz = xdx + ydy + \frac{a}{2}(dx + dy) \pm \frac{1}{2}\sqrt{\{2(x-y)^2 - a^2\}}(dx - dy)$$

Integrating, the required complete integral is

$$z = \frac{x^2 + y^2}{2} + \frac{a(x+y)}{2} \pm \frac{1}{\sqrt{2}}\left\{\frac{x-y}{2}\sqrt{(x-y)^2 - \frac{a^2}{2}} - \frac{a^2}{4}\log\left[(x-y) + \sqrt{(x-y)^2 - \frac{a^2}{2}}\right]\right\} + b$$

where a, b are arbitrary constants.

17 ■▶ ***Find the complete integral of $q = (z + px)^2$ by Charpit's method.***

The given equation is

$$f(x, y, z, p, q) \equiv (z + px)^2 - q = 0 \text{ .... (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \text{ .... (2)}$$

Here $f_x = 2p(z + px), f_y = 0, f_z = 2(z + px), f_p = 2x\,(z + px), f_q = -1$.

$\therefore$ From (2) we get,

$$\frac{dp}{2p(z+px) + 2p(z+px)} = \frac{dq}{2q(z+px)} = \frac{dz}{-2xp(z+px) + q} = \frac{dx}{-2x(z+px)} = \frac{dy}{1} \text{ .... (3)}$$

From 2nd and 4th ratio of (3) we get

$$\frac{dq}{q} = \frac{-dx}{x}. \text{ Integrating } \log q = \log a - \log x \text{ or, } q = \frac{a}{x} \text{ .... (4)}$$

Putting the value of q in (1) we get

$$(z + px)^2 = \frac{a}{x} \quad \text{or, } z + px = \sqrt{\frac{a}{x}}$$

$$\text{or, } \quad px = \frac{\sqrt{a}}{\sqrt{x}} - z \text{ or, } p = \frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}.$$

$$\text{or, } \quad px = \frac{\sqrt{a}}{\sqrt{x}} - z \text{ or, } p = \frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}.$$

Putting the value of p and q in $dz = px + qdy$ we get,

$$dz = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right)dx + \frac{a}{x}dy$$

$$\text{or, } \quad xdz = \sqrt{a}x^{-\frac{1}{2}}dx - zdx + ady \text{ or, } xdz + zdx = \sqrt{a}x^{-\frac{1}{2}}dx + ady$$

$$\text{or, } \quad d(xz) = \sqrt{a}x^{-\frac{1}{2}}dx + ady$$

Integrating, $xz = 2\sqrt{a}\sqrt{x} + ay + b$, where a, b are arbitrary constants.
which is the required complete integral.

18 ■▶ ***Find the complete integral of $2z + p^2 + qy + 2y^2 = 0$ by using Charpit's method.***

The given equation is

$$f(x, y, z, p, q) \equiv 2z + p^2 + qy + 2y^2 = 0 \text{ .... (1)}$$

Charpit's auxliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{..............} (2)$$

Here $f_x = 0, f_y = q + 4y, f_z = 2, f_p = 2p, f_q = y$

$\therefore$ From (2) we get

$$\frac{dp}{2p} = \frac{dq}{q+4y+2q} = \frac{dz}{-2p^2 - qy} = \frac{dp}{-2p} = \frac{dq}{-y}$$

or, $$\frac{dp}{2p} = \frac{dq}{3q+4y} = \frac{dz}{-2p^2 - qy} = \frac{dx}{-2p} = \frac{dy}{-y}.$$

next do by yourself.....................