LECTURE NOTE-3

Solution of Linear PARTIAL DIFFERENTIAL EQUATIONS

LAGRANGE'S METHOD:

An equation of the form $Pp + Qq = R$ is said to be Lagrange's type of partial differential equations.

Working Rule for solving $Pp + Qq = R$ by Lagrange's method

STEP 1 Put the given linear partial differential equation of first order in the standard from

$$Pp + Qq = R \text{ .... } (1)$$

STEP 2 Write down Lagrange's auxliary equation for (1) namely

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \text{ .... } (2)$$

STEP 3 Solve (2) by using the well known methods. Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

STEP 4 The general solution (or integral) of (1) is then written in one of the following three equivalent forms :

$\varphi(u, v) = 0$, $u = \varphi(v)$ or, $v = \varphi(u)$, φ being an arbitrary function.

Examples

1.5.(a) Type I for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Ex.1. Solve $a(p + q) = z$

Solution: Given $ap + aq = z$ ...(1.29)

The Lagrange's Auxiliary equation of (1.29) are

$$\frac{dx}{a} = \frac{dy}{a} = \frac{dz}{1} \quad \text{....(1.30)}$$

Taking the first two members of (1.30) we have

$$dx - dy = 0 \quad \text{...(1.31)}$$

Integrating (1.31)

$x - y = c_1$ ...(1.32)

Taking the last two members of (1.30) we have

$$dy - a\,dz = 0 \qquad \text{...(1.33)}$$

Integrating (1.33) ,

$$y - az = c2 \qquad \text{...(1.34)}$$

From (1.32) and (1.34), the required solution is given by

ϕ(x- y, y – az) = 0, ϕ being an arbitrary function.

Exercise 2. Solve 2p + 3q =1

Solution. Given 2p + 3q =1 ....(1.35)

The Lagrange's Auxiliary equation of (1.35) are

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1} \qquad \text{...(1.36)}$$

Taking the first two members of (1.36), we have

$$3dx - 2dy = 0 \qquad \text{...(1.37)}$$

Integrating (1.37)

$$3x - 2y = c_1 \qquad \text{...(1.38)}$$

Taking the last two members of (1.36), we have

$$dy - 3dz = 0 \qquad \text{...(1.39)}$$

Integrating (1.39)

$$y - 3z = c_2 \qquad \text{...(1.40)}$$

From (1.38) and (1.40), the required solution is given by

$\phi(3x - 2y, y - 3z) = 0$, ϕ being an arbitrary function.

Ex. 3. Solve $yzp + 2xq = xy$

1.5 (b) Type 2 for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Ex.1. Solve $xzp + yzq = xy$

Sol. Given $xzp + yzq = xy$ ...(1.41)

The Lagrange's subsidiary equations for (1.41) are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \qquad ...(1.42)$$

Taking the first two fractions of (1.42)

$$\frac{dx}{x} - \frac{dy}{y} = 0 \qquad ...(1.43)$$

Integrating (1.43),

$$\log x - \log y = \log c_1 \quad or \quad x/y = c_1 \qquad ...(1.44)$$

From (1.44) $x = c_1 y$.

From second and third fraction of (1.42), we get

$$\frac{dy}{yz} = \frac{dz}{c_1 y^2} \quad \text{or} \quad c_1 y dy - z dz = 0 \qquad ...(1.45)$$

Integrating (1.45)

$$\tfrac{1}{2} c_1 y^2 - \tfrac{1}{2} z^2 = \tfrac{1}{2} c_2$$

or $$c_1 y^2 - z^2 = c_2$$

or $$xy - z^2 = c_2 \quad \text{using (1.44)} \qquad ...(1.46)$$

From (1.44) and (1.46), the required general solution is

$$\phi(xy - z^2, x/y) = 0, \ \phi \text{ being an arbitrary function.}$$

Exercise 2. Solve the partial differential equation $p - 2q = 3x^2 \sin(y+2x)$

1.5(c) Type 3 for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Ex 1. Solve $(mz - ny)p + (nx - lz)q = ly - mx$

Sol. The Lagrange's auxiliary equation of the given equation are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \qquad \text{...(1.47)}$$

Changing x, y, z as multipliers, each fraction of (1.47)

$$= \frac{xdx + ydy + zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

Therefore $\quad xdx + ydy + zdz = 0$

or $\quad 2xdx + 2ydy + 2zdz = 0$

Integrating, $\quad x^2 + y^2 + z^2 = c_1 \quad .\qquad$ .. (1.48)

Again choose l, m, n as multipliers, each fraction of (1.47)

$$= \frac{ldx + mdy + ndz}{l(mz-ny) + m(nx-lz) + n(ly-mx)}$$

$$= \frac{ldx + mdy + n\,dz}{0}$$

Therefore $\quad ldx + mdy + n\,dz = 0$

so that $\quad lx + my + nz = c_2 \qquad$ …(1.49)

From (1.48) and (1.49), the required general solution is given by

$$\phi(x^2 + y^2 + z^2,\ ldx + mdy + n\,dz) = 0$$

Exercise 2. Solve $x(y-z)p + y(z-x)\,q = \frac{(x-y)}{xy}$

Solve the following partial differnetial equations:

(1). Solve $pz - qz = z^2 + (x+y)^2$

Solution: $pz - qz = z^2 + (x+y)^2 \qquad$ …(1.52)

The Lagrange's subsidiary equations are

$$\frac{dx}{z} = \frac{dy}{z} = \frac{dz}{z^2+(x+y)^2}$$

Taking the first two fractions of (1.52), we get

$$x + y = c_1$$

From first and third fraction of (1.52), we get

$$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$$

or $$\frac{dx}{z} = \frac{dz}{z^2+c_1^2}$$

Integrating we get

or $$\int dx = \frac{1}{2}\int \frac{2zdz}{z^2+c_1^2}$$

or $$x = \frac{1}{2}\log(z^2 + c_1^2) + c_2$$

or $$x - \frac{1}{2}\log(z^2 + c_1^2) = c_2$$

The general solutions is given by

$$f(c_1, c_2) = 0$$

or $$f\left[x + y,\ x - \frac{1}{2}\log\{z^2 + (x + y)^2\}\right] = 0$$

Exercise (2). Solve $(y + z)p + (z + x)q = (x + y)$

Sol: $(y + z)p + (z + x)q = (x + y)$

The auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad \ldots (1.53)$$

Using $(1, -1, 0)$ and $(1, 0, -1)$ each fraction of (1.53) is equal to

$$\frac{dx-dy}{y+z-z-x} = \frac{dx-dz}{y+z-x-y}$$

$$\frac{dx-dy}{y-x} = \frac{dx-dz}{z-x}$$

Integrating we get

or $$\int \frac{dx-dy}{y-x} = \int \frac{dx-dz}{z-x}$$

$$\log(x-y) = \log(x-z) + \log c_1$$

or $$\frac{x-y}{x-z} = c_1$$

Also using multipliers $(1,-1,0)$ and $(1,1,1)$, each fraction of (1.53) is equal to

or $$\frac{dx-dy}{y+z-z-x} = \frac{dx+dy+dz}{2(x+y+z)}$$

or $$2\frac{dx-dy}{x-y} = -\frac{dx+dy+dz}{(x+y+z)}$$

Integrating we get

$$2\log(x-y) = -\log(x+y+z) + \log c_2$$

or $$(x-y)^2(x+y+z) = c_2$$

The solution of the given equation is

$$f(c_1, c_2) = 0$$

or $$f\left[\frac{x-y}{x-z},\ (x-y)^2(x+y+z)\right] = 0$$

Exercise (3). Solve $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

Solution: $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

The subsidiary auxiliary equations are

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{(xy+zx)} = \frac{dz}{xy-zx} \quad \ldots(1.54)$$

From 2nd and 3rd fraction we get

$$\frac{dy}{xy+zx} = \frac{dz}{xy-zx}$$

or $$\frac{dy}{dz} = \frac{y+z}{y-z} \quad \ldots(1.55)$$

It is a homogeneous equation therefore we put $y = vz$

So that $$\frac{dy}{dz} = v + z\frac{dv}{dz}$$

Then (1.55) reduces to

$$v+z\frac{dv}{dz}=\frac{(vz+z)}{(vz-z)}$$

$$\Rightarrow \qquad z\frac{dv}{dz}=\frac{v+1}{v-1}-v$$

$$\Rightarrow \qquad z\frac{dv}{dz}=\frac{v+1-v^2+v}{v-1}$$

$$\Rightarrow \qquad z\frac{dv}{dz}=\frac{-v^2+2v+1}{v-1}$$

$$\Rightarrow \qquad \frac{(v-1)dv}{v^2-2v-1}=-\frac{dz}{z}$$

$$\Rightarrow \qquad -\frac{(v-1)dv}{v^2-2v-1}=\frac{dz}{z}$$

Integrating we get

$$\Rightarrow \qquad -\frac{1}{2}\int\frac{2(v-1)dv}{v^2-2v-1}=\int\frac{dz}{z}$$

$$\Rightarrow \qquad -\frac{1}{2}\log\left(v^2-2v-1\right)=\log z+\log c_1$$

$$\Rightarrow \qquad \left(v^2-2v-1\right)^{-\frac{1}{2}}=zc_1$$

$$\Rightarrow \qquad c_1=\frac{1}{\left(z^2v^2-2z^2v-z^2\right)^{\frac{1}{2}}}$$

$$\Rightarrow \qquad c_1=\frac{1}{\left(z^2\frac{y^2}{z^2}-2z^2\frac{y}{z}-z^2\right)^{\frac{1}{2}}}$$

Also using multipliers (x, y, z), we get fraction of (1.55), equal to

$$\frac{xdx+ydy+zdz}{xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2}$$

or $$\frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow \qquad xdx + ydy + zdz = 0$$

Integrating we get

$$\Rightarrow \qquad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$$\Rightarrow \qquad x^2 + y^2 + z^2 = c_2$$

The general solution is given by

$$f(c_1, c_2) = 0$$

or
$$f\left(\frac{1}{\left(y^2 - 2zy - z^2\right)^{\frac{1}{2}}}, x^2 + y^2 + z^2\right) = 0$$

Exercise (4). Solve $(y^2 + z^2 - x^2)p + 2xyq + 2zx = 0$.

Exercise (5). Solve $\left(\frac{y^2 z}{x}\right)p + zxq = y^2$

Solution: The Lagrange's auxiliary equations are

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{zx} = \frac{dz}{y^2}$$

or
$$\frac{xdx}{y^2 z} = \frac{dy}{zx} = \frac{dz}{y^2}$$

From 1st and 2nd fractions we get

$$\frac{xdx}{y^2 z} = \frac{dy}{zx}$$

or
$$x^2 dx = y^2 dy$$

integrating

$$\frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3}$$

$$c_1 = x^3 - y^3$$

From 1st and 3rd fractions we get

$$\frac{xdx}{y^2 z} = \frac{dz}{y^2}$$

or $$xdx = zdz$$

integrating

or $$\frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

or $$c_2 = x^2 - z^2$$

The general solution is given by

$$f(c_1, c_2) = 0$$

$$f(x^3 - y^3, x^2 - z^2) = 0$$

Exercise (6). Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Solution: The subsidiary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

Using the multipliers $(1, -1, 0)$ and $(0, 1, -1)$, we get

$$\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy}$$

$$\Rightarrow \quad \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{y^2-z^2+x(y-z)}$$

$$\Rightarrow \quad \frac{dx-dy}{(x-y)(x+y)+z(x-y)} = \frac{dy-dz}{(y-z)(y+z)+x(y-z)}$$

$$\Rightarrow \quad \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$$\Rightarrow \quad \frac{dx-dy}{(x-y)} = \frac{dy-dz}{(y-z)}$$

Integrating we get

$$log\,(x-y) = \log(y-z) + logc_1$$

$$\Rightarrow \quad c_1 = \frac{x-y}{y-z}$$

Using the multipliers $(1, 0, -1)$ and $(0, 1, -1)$, we get

$$\frac{dx-dz}{x^2-yz-z^2+xy} = \frac{dy-dz}{y^2-zx-z^2+xy}$$

$\Rightarrow$ $$\frac{dx-dz}{x^2-z^2+y(x-z)} = \frac{dy-dz}{y^2-z^2+x(y-z)}$$

$\Rightarrow$ $$\frac{dx-dz}{(x-z)(x+z)+y(x-z)} = \frac{dy-dz}{(y-z)(y+z)+x(y-z)}$$

$\Rightarrow$ $$\frac{dx-dz}{(x-z)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$\Rightarrow$ $$\frac{dx-dz}{(x-z)} = \frac{dy-dz}{(y-z)}$$

Integrating we get

$$\log\,(x-z) = \log(y-z) + \log\, c_2$$

$\Rightarrow$ $$c_2 = \frac{x-z}{y-z}$$

The general solution is $f(c_1, c_2) = 0$

or $$f\left(\frac{x-y}{y-z}, \frac{x-z}{y-z}\right) = 0$$

Exercise (7) $\quad z(xp - yq) = y^2 - x^2$

Exercise (8). $\quad \frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$

Solution: Here $\quad P = \frac{y-z}{yz}, \quad Q = \frac{z-x}{zx}, \quad R = \frac{x-y}{xy}$

The Lagrange's auxiliary equations are

$$\frac{dx}{\frac{y-z}{yz}} = \frac{dy}{\frac{z-x}{zx}} = \frac{dz}{\frac{x-y}{xy}}$$

$\Rightarrow$ $$\frac{yzdx}{y-z} = \frac{zxdy}{z-x} = \frac{xydz}{x-y}$$

Using multipliers (x, y, z), each ratio is equal to

$$\frac{xyzdx+xyzdy+xyzdz}{0}$$

$\Rightarrow$ $$dx + dy + dz = 0$$

$\Rightarrow$ $$x + y + z = c_1$$

Also using $(1,1,1)$, we have each ratio equal to

$$\frac{yzdx+xzdy+xydz}{0}$$

$\Rightarrow$ $$yzdx + xzdy + xydz = 0$$

$\Rightarrow$ $$d(xyz) = 0$$

$\Rightarrow$ $$xyz = c_2$$

The general solution is $f(c_1, c_2) = 0$

or $$f(x + y + z, xyz) = 0$$

Exercise (9) $$z - xp - yq = a\sqrt{x^2 + y^2 + z^2}$$

Sol: The given equation can be written as

$$xp + yq = z - a\sqrt{x^2 + y^2 + z^2}$$

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-a\sqrt{x^2+y^2+z^2}} \qquad \ldots (1.56)$$

Using (x, y, z) as multipliers and equating the resultant fraction with the first fraction of (1.56) we get

$$\frac{xdx+ydy+zdz}{x^2+y^2+z^2-az\sqrt{x^2+y^2+z^2}} = \frac{dx}{x} \qquad \ldots (1.57)$$

Taking $x^2 + y^2 + z^2 = u^2$; we have

$$xdx + ydy + zdz = du$$

So that equation (1.57) reduces to

$$\frac{du}{u-az} = \frac{dx}{x}$$

Integrating, we get

$$\log(u - az) = \log(x) + \log c_1$$

$\Rightarrow$ $$\frac{u-az}{x} = c_1$$

or $$\frac{\sqrt{x^2+y^2+z^2}-az}{x} = c_1$$

Now taking the 1st two fractions of (1.56), we get

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$\Rightarrow$ $$\log x = \log y + \log c_2$$

$\Rightarrow$ $$\frac{x}{y} = c_2$$

The general solution of the given equation is given by $f(c_1, c_2) = 0$

or $$f\left(\frac{\sqrt{x^2+y^2+z^2}-az}{x}, \frac{x}{y}\right)$$

Exercise (10): **Solve** $p + 3q = 5z + \tan(y - 3x)$

Solution: Here $P = 1, \quad Q = 3, \ and\ R = 5z + \tan(y - 3x)$

Therefore the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)} \qquad \ldots(1.58)$$

From first two fractions of (1) we get

$$\frac{dx}{1} = \frac{dy}{3}$$

$\Rightarrow$ $$dy - 3dx = 0$$

$\Rightarrow$ $$y - 3x = c_1$$

From 1st and 3rd fractions of (1.58) we get

$$\frac{dx}{1} = \frac{dz}{5z+\tan(y-3x)}$$

$\Rightarrow$ $$\frac{dx}{1} = \frac{dz}{5z+\tan c_1}$$ [using c_1]

Integrating, we get

$\Rightarrow$ $$\int dx = \int \frac{dz}{5z+\tan c_1}$$

$\Rightarrow \qquad x = \frac{1}{5}\log(5z + \tan c_1) + c_2$

$\Rightarrow \qquad x - \frac{1}{5}\log\{5z + \tan(y - 3x)\} = c_2 \qquad \dots (1.59)$

The general solution is given by $f(c_1, c_2) = 0$

or $\qquad f\left[y - 3x,\ x - \frac{1}{5}\log\{5z + \tan(y - 3x)\}\right] = 0$