LECTURE NOTE - 5

The matrix method for homogeneous linear systems with constant coefficients: n Equations in n Unknowns:

Theorem 7.16:

Consider the homogeneous linear system

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,\\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n,\\ &\vdots\\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n,\end{aligned} \tag{7.136}$$

that is, the vector differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \tag{7.139}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

and the a_{ij}, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n)$, are real constants.

Suppose each of the n characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ is distinct; and let $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \ldots, \boldsymbol{\alpha}^{(n)}$ be a set of n respective corresponding characteristic vectors of $\mathbf{A}$.

Then on every real interval, the n vector functions defined by

$$\boldsymbol{\alpha}^{(1)}e^{\lambda_1 t}, \boldsymbol{\alpha}^{(2)}e^{\lambda_2 t}, \ldots, \boldsymbol{\alpha}^{(n)}e^{\lambda_n t}$$

form a linearly independent set of solutions of (7.136), that is, (7.139); and

$$\mathbf{x} = c_1\boldsymbol{\alpha}^{(1)}e^{\lambda_1 t} + c_2\boldsymbol{\alpha}^{(2)}e^{\lambda_2 t} + \cdots + c_n\boldsymbol{\alpha}^{(n)}e^{\lambda_n t},$$

where $c_1, c_2, \ldots, c_n$ are n arbitrary constants, is a general solution of (7.136).

Example 1:

Consider the homogeneous linear system

$$\frac{dx_1}{dt} = 7x_1 - x_2 + 6x_3,$$

$$\frac{dx_2}{dt} = -10x_1 + 4x_2 - 12x_3, \tag{7.150}$$

$$\frac{dx_3}{dt} = -2x_1 + x_2 - x_3,$$

that is, the vector differential equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{7.151}$$

Assuming a solution of (7.151) of the form

$$\mathbf{x} = \boldsymbol{\alpha} e^{\lambda t},$$

that is,

$$x_1 = \alpha_1 e^{\lambda t}, \quad x_2 = \alpha_2 e^{\lambda t}, \quad x_3 = \alpha_3 e^{\lambda t},$$

we know that λ must be a solution of the characteristic equation of the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}.$$

This characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, we see that the characteristic equation is

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0,$$

the factored form of which is

$$(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0.$$

We thus see that the characteristic values of $\mathbf{A}$ are

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \text{and} \quad \lambda_3 = 5.$$

These are distinct (and real), and so Theorem 7.16 applies. We thus proceed to find characteristic vectors $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \boldsymbol{\alpha}^{(3)}$ corresponding respectively to $\lambda_1, \lambda_2, \lambda_3$. We use the defining equation

$$\mathbf{A}\boldsymbol{\alpha} = \lambda\boldsymbol{\alpha} \tag{7.145}$$

to do this.

For $\lambda = \lambda_1 = 2$ and

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix},$$

defining equation (7.145) becomes

$$\begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 2\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Performing the indicated multiplications and equating corresponding components of this, we find

$$\begin{aligned} 7\alpha_1 - \alpha_2 + 6\alpha_3 &= 2\alpha_1, \\ -10\alpha_1 + 4\alpha_2 - 12\alpha_3 &= 2\alpha_2, \\ -2\alpha_1 + \alpha_2 - \alpha_3 &= 2\alpha_3. \end{aligned}$$

Simplifying, we find that $\alpha_1, \alpha_2, \alpha_3$ must satisfy

$$\begin{aligned} 5\alpha_1 - \alpha_2 + 6\alpha_3 &= 0, \\ -10\alpha_1 + 2\alpha_2 - 12\alpha_3 &= 0, \\ -2\alpha_1 + \alpha_2 - 3\alpha_3 &= 0, \end{aligned} \tag{7.152}$$

The second of these three equations is merely a constant multiple of the first. Thus we seek nonzero numbers $\alpha_1, \alpha_2, \alpha_3$ that satisfy the first and third of these equations. Writing these two as equations in the unknowns α_2 and α_3, we have

$$\begin{aligned} -\alpha_2 + 6\alpha_3 &= -5\alpha_1, \\ \alpha_2 - 3\alpha_3 &= 2\alpha_1. \end{aligned}$$

Solving for α_2 and α_3, we find

$$\alpha_2 = -\alpha_1 \quad \text{and} \quad \alpha_3 = -\alpha_1.$$

A simple nontrivial solution of this is $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = -1$. That is, $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = -1$ is a simple nontrivial solution of the system (7.152). Thus a characteristic vector corresponding to $\lambda_1 = 2$ is

$$\alpha^{(1)} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Then by Theorem 7.16,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{2t}, \quad \text{that is,} \quad \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}, \tag{7.153}$$

is a solution of (7.151).

For $\lambda = \lambda_2 = 3$ and

$$\alpha = \alpha^{(2)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix},$$

(7.145) becomes

$$\begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 3 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Performing the indicated multiplications and equating corresponding components of this and then simplifying, we find that $\alpha_1, \alpha_2, \alpha_3$ must satisfy

$$\begin{aligned} 4\alpha_1 - \alpha_2 + 6\alpha_3 &= 0, \\ -10\alpha_1 + \alpha_2 - 12\alpha_3 &= 0, \\ -2\alpha_1 + \alpha_2 - 4\alpha_3 &= 0. \end{aligned}$$

From these we find that

$$\alpha_2 = -2\alpha_1 \quad \text{and} \quad \alpha_3 = -\alpha_1.$$

A simple nontrivial solution of this is $\alpha_1 = 1, \alpha_2 = -2, \alpha_3 = -1$. Thus a characteristic vector corresponding to $\lambda_2 = 3$ is

$$\alpha^{(2)} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Then by Theorem 7.16,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{3t}, \quad \text{that is,} \quad \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}, \tag{7.154}$$

is a solution of (7.151).

For $\lambda = \lambda_3 = 5$ and

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(3)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix},$$

(7.145) becomes

$$\begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 5 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Performing the indicated multiplications and equating corresponding components of this, and then simplifying, we find that $\alpha_1, \alpha_2, \alpha_3$ must satisfy

$$\begin{aligned} 2\alpha_1 - \alpha_2 + 6\alpha_3 &= 0, \\ -10\alpha_1 - \alpha_2 - 12\alpha_3 &= 0, \\ -2\alpha_1 + \alpha_2 - 6\alpha_3 &= 0. \end{aligned}$$

From these we find that

$$\alpha_2 = -2\alpha_1 \quad \text{and} \quad 3\alpha_3 = -2\alpha_1.$$

A simple nontrivial solution of this is $\alpha_1 = 3, \alpha_2 = -6, \alpha_3 = -2$. Thus a characteristic vector corresponding to $\lambda_3 = 5$ is

$$\boldsymbol{\alpha}^{(3)} = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}.$$

Then by Theorem 7.16,

$$\mathbf{x} = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} e^{5t}, \quad \text{that is,} \quad \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix} \tag{7.155}$$

is a solution of (7.151).

Also by Theorem 7.16, the solutions (7.153), (7.154), and (7.155) are linearly independent, and a general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} + c_3 \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix},$$

where c_1, c_2, and c_3 are arbitrary constants. That is, in scalar language, a general solution of the homogeneous linear system (7.150) is

$$x_1 = c_1 e^{2t} + c_2 e^{3t} + 3c_3 e^{5t},$$
$$x_2 = -c_1 e^{2t} - 2c_2 e^{3t} - 6c_3 e^{5t},$$
$$x_3 = -c_1 e^{2t} - c_2 e^{3t} - 2c_3 e^{5t},$$

where c_1, c_2, and c_3 are arbitrary constants.

We return to the homogeneous linear system (7.136), that is, the vector differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \tag{7.139}$$

where $\mathbf{A}$ is an $n \times n$ real constant matrix, and reconsider the result stated in Theorem 7.16. In that theorem we stated that if each of the n characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ is *distinct* and if $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \ldots, \boldsymbol{\alpha}^{(n)}$ is a set of n respective corresponding characteristic vectors of $\mathbf{A}$, then the n functions defined by

$$\boldsymbol{\alpha}^{(1)} e^{\lambda_1 t}, \boldsymbol{\alpha}^{(2)} e^{\lambda_2 t}, \ldots, \boldsymbol{\alpha}^{(n)} e^{\lambda_n t}$$

form a fundamental set of solutions of (7.139). Note that although we assume that λ_1, $\lambda_2, \ldots, \lambda_n$ are *distinct*, we do *not* require that they be *real*. Thus distinct *complex* characteristic values may be present. However, since $\mathbf{A}$ is a real matrix, any complex characteristic values must occur in conjugate pairs. Suppose $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ form such a pair. Then the corresponding solutions are

$$\boldsymbol{\alpha}^{(1)} e^{(a+bi)t} \quad \text{and} \quad \boldsymbol{\alpha}^{(2)} e^{(a-bi)t},$$

and these solutions are *complex* solutions. Thus if one or more distinct conjugate-complex pairs of characteristic values occur, the fundamental set defined by $\boldsymbol{\alpha}^{(i)} e^{\lambda_i t}$, $i = 1, 2, \ldots, n$, contains *complex* functions. However, in such a case, this fundamental set may be replaced by another fundamental set, all of whose members are *real* function.

Case of Repeated Characteristic Values

We again consider the vector differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \tag{7.139}$$

where $\mathbf{A}$ is an $n \times n$ real constant matrix; but here we give an introduction to the case in which $\mathbf{A}$ has a repeated characteristic value. To be definite, we suppose that $\mathbf{A}$ has a real characteristic value λ_1 of multiplicity m, where $1 < m \leq n$, and that all the other characteristic values $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$ (if there are any) are distinct. By result G of

Section 7.5D, we know that the repeated characteristic value λ_1 of multiplicity m has p linearly independent characteristic vectors, where $1 \leq p \leq m$. Now consider two subcases: (1) $p = m$; and (2) $p < m$.

In Subcase (1), $p = m$, there are m linearly independent characteristic vectors $\alpha^{(1)}$, $\alpha^{(2)}, \ldots, \alpha^{(m)}$ corresponding to the characteristic value λ_1 of multiplicity m. Then the n functions defined by

$$\alpha^{(1)}e^{\lambda_1 t}, \alpha^{(2)}e^{\lambda_1 t}, \ldots, \alpha^{(m)}e^{\lambda_1 t}, \alpha^{(m+1)}e^{\lambda_{m+1} t}, \ldots, \alpha^{(n)}e^{\lambda_n t}$$

form a linearly independent set of n solutions of differential equation (7.139); and a general solution of (7.139) is a linear combination of these n solutions having n arbitrary numbers as the "constants of combination."

EXAMPLE-2:

Consider the homogeneous linear system

$$\begin{aligned}
\frac{dx_1}{dt} &= 3x_1 + x_2 - x_3,\\
\frac{dx_2}{dt} &= x_1 + 3x_2 - x_3, \qquad (7.156)\\
\frac{dx_3}{dt} &= 3x_1 + 3x_2 - x_3,
\end{aligned}$$

or in matrix form,

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix}\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \qquad (7.157)$$

Assuming a solution of the form

$$\mathbf{x} = \boldsymbol{\alpha} e^{\lambda t},$$

that is,

$$x_1 = \alpha_1 e^{\lambda t}, \quad x_2 = \alpha_2 e^{\lambda t}, \quad x_3 = \alpha_3 e^{\lambda t},$$

we know that λ must be a solution of the characteristic equation of the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix}.$$

This characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, we see that the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0,$$

the factored form of which is

$$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0.$$

We thus see that the characteristic values of $\mathbf{A}$ are

$$\lambda_1 = 1, \quad \lambda_2 = 2 \quad \text{and} \quad \lambda_3 = 2.$$

Note that whereas the number 1 is a *distinct* characteristic value of $\mathbf{A}$, the number 2 is a *repeated* characteristic value. We again use

$$\mathbf{A}\boldsymbol{\alpha} = \lambda\boldsymbol{\alpha} \tag{7.145}$$

to find characteristic vectors corresponding to these characteristic values.

For $\lambda = 1$, and

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix},$$

(7.145) becomes

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 1 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Performing the indicated multiplications and equating corresponding components of this and then simplifying, we find that $\alpha_1, \alpha_2, \alpha_3$ must be a nontrivial solution of the system

$$\begin{aligned} 2\alpha_1 + \alpha_2 - \alpha_3 &= 0, \\ \alpha_1 + 2\alpha_2 - \alpha_3 &= 0, \\ 3\alpha_1 + 3\alpha_2 - 2\alpha_3 &= 0. \end{aligned}$$

One readily sees that such a solution is given by

$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 3.$$

Thus a characteristic vector corresponding to $\lambda_1 = 1$ is

$$\boldsymbol{\alpha}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

Then

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} e^t, \quad \text{that is,} \quad \begin{pmatrix} e^t \\ e^t \\ 3e^t \end{pmatrix}, \tag{7.158}$$

is a solution of (7.157).

We now turn to the repeated characteristic value $\lambda_2 = \lambda_3 = 2$. In terms of the discussion just preceding this example, this characteristic value 2 has multiplicity $m = 2 < 3 = n$, where $n = 3$ is the common number of rows and columns of the coefficient matrix $\mathbf{A}$. For $\lambda = 2$ and

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \tag{7.159}$$

(7.145) becomes

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 2 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Equating corresponding coefficients of this and simplifying, we find that $\alpha_1, \alpha_2, \alpha_3$ must be a nontrivial solution of the system

$$\begin{aligned} \alpha_1 + \alpha_2 - \alpha_3 &= 0, \\ \alpha_1 + \alpha_2 - \alpha_3 &= 0, \\ 3\alpha_1 + 3\alpha_2 - 3\alpha_3 &= 0. \end{aligned}$$

Note that each of these three relations is equivalent to each of the other two, and so the only relationship among $\alpha_1, \alpha_2, \alpha_3$ is that given most simply by

$$\alpha_1 + \alpha_2 - \alpha_3 = 0. \tag{7.160}$$

Observe that

$$\alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = 0$$

and

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = 1$$

are two distinct solutions of this relation (7.160). The corresponding vectors of the form (7.159) are thus

$$\boldsymbol{\alpha}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\alpha}^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

respectively. Since each satisfies (7.145) with $\lambda = 2$, each is a characteristic vector corresponding to the double root $\lambda_2 = \lambda_3 = 2$. Furthermore, using the definition of linear independence of a set of constant vectors, one sees that these vectors $\boldsymbol{\alpha}^{(2)}$ and $\boldsymbol{\alpha}^{(3)}$ are linearly independent. Thus the characteristic value $\lambda = 2$ of multiplicity $m = 2$ has the $p = 2$ linearly independent characteristic vectors

$$\boldsymbol{\alpha}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\alpha}^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

corresponding to it. Hence this is an illustration of Subcase (1) of the discussion preceding this example. Thus, corresponding to the twofold characteristic value $\lambda = 2$, there are two linearly independent solutions of system (7.157) of the form $\boldsymbol{\alpha} e^{\lambda t}$. These are

$$\boldsymbol{\alpha}^{(2)} e^{2t} \quad \text{and} \quad \boldsymbol{\alpha}^{(3)} e^{2t},$$

that is,

that is,

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t},$$

or

$$\begin{pmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}, \tag{7.161}$$

respectively.

The three solutions

$$\begin{pmatrix} e^{t} \\ e^{t} \\ 3e^{t} \end{pmatrix}, \quad \begin{pmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$$

given by (7.158) and (7.161) are linearly independent, and a general solution is

$$\mathbf{x} = c_1\begin{pmatrix} e^t \\ e^t \\ 3e^t \end{pmatrix} + c_2\begin{pmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{pmatrix} + c_3\begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix},$$

where c_1, c_2, and c_3 are arbitrary constants. That is, in scalar language, a general solution of the homogeneous linear system (7.156) is

$$\begin{aligned} x_1 &= c_1 e^t + (c_2 + c_3)e^{2t}, \\ x_2 &= c_1 e^t - c_2 e^{2t}, \\ x_3 &= 3c_1 e^t + c_3 e^{2t}. \end{aligned}$$

where c_1, c_2, and c_3 are arbitrary numbers.

One type of vector differential equation (7.139) which *always* leads to Subcase (1), $p = m$, in the case of a repeated characteristic value λ_1 is that in which the $n \times n$ coefficient matrix $\mathbf{A}$ of (7.139) is a real symmetric matrix. For then, by Result J of Section 7.5D, there always exist n linearly independent characteristic vectors of $\mathbf{A}$, regardless of whether the n characteristic values of $\mathbf{A}$ are all distinct or not.

We now turn to a consideration of Subcase (2), $p < m$. In this case, there are less than m linearly independent characteristic vectors $\boldsymbol{\alpha}^{(1)}$ corresponding to the characteristic value λ_1 of multiplicity m. Hence there are less than m linearly independent solutions of system (7.136) of the form $\boldsymbol{\alpha}^{(1)}e^{\lambda_1 t}$ corresponding to λ_1. Thus there is *not* a full set of n linearly independent solutions of (7.136) of the basic exponential form $\boldsymbol{\alpha}^{(k)}e^{\lambda_k t}$, where λ_k is a characteristic value of $\mathbf{A}$ and $\boldsymbol{\alpha}^{(k)}$ is a characteristic vector corresponding to λ_k. Clearly we must seek linearly independent solutions of another form.

To discover what other forms of solution to seek, we first look back at the analogous situation in Section 7.6C. The results there suggest the following:

Let λ be a characteristic value of multiplicity $m = 2$. Suppose $p = 1 < m$, so that there is only one type of characteristic vector $\boldsymbol{\alpha}$ and hence only one type of solution of the basic exponential form $\boldsymbol{\alpha}e^{\lambda t}$ corresponding to λ. Then a linearly independent solution is of the form

$$(\boldsymbol{\alpha}t + \boldsymbol{\beta})e^{\lambda t},$$

where $\boldsymbol{\alpha}$ is a characteristic vector corresponding to λ, that is, $\boldsymbol{\alpha}$ satisfies

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\alpha} = \mathbf{0};$$

and $\boldsymbol{\beta}$ is a vector which satisfies the equation

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\beta} = \boldsymbol{\alpha}.$$

Now let λ be a characteristic value of multiplicity $m = 3$ and suppose $p < m$. Here there are two possibilities: $p = 1$ and $p = 2$.

If $p = 1$, there is only one type of characteristic vector $\boldsymbol{\alpha}$ and hence only one type of solution of the form

$$\boldsymbol{\alpha} e^{\lambda t} \tag{7.162}$$

corresponding to λ. Then a second solution corresponding to λ is of the form

$$(\boldsymbol{\alpha} t + \boldsymbol{\beta}) e^{\lambda t}, \tag{7.163}$$

where $\boldsymbol{\alpha}$ is a characteristic value corresponding to λ, that is, $\boldsymbol{\alpha}$ satisfies

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{\alpha} = \mathbf{0}; \tag{7.164}$$

and $\boldsymbol{\beta}$ is a vector which satisfies the equation

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{\beta} = \boldsymbol{\alpha}. \tag{7.165}$$

In this case, a third solution corresponding to λ is of the form

$$\left(\boldsymbol{\alpha} \frac{t^2}{2!} + \boldsymbol{\beta} t + \boldsymbol{\gamma}\right) e^{\lambda t}, \tag{7.166}$$

where $\boldsymbol{\alpha}$ satisfies (7.164), $\boldsymbol{\beta}$ satisfies (7.165), and γ satisfies

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{\gamma} = \boldsymbol{\beta}. \tag{7.167}$$

The three solutions (7.162), (7.163), and (7.166) so found are linearly independent.

If $p = 2$, there are two linearly independent characteristic vectors $\boldsymbol{\alpha}^{(1)}$ and $\boldsymbol{\alpha}^{(2)}$ corresponding to λ and hence there are two linearly independent solutions of the form

$$\boldsymbol{\alpha}^{(1)} e^{\lambda t} \quad \text{and} \quad \boldsymbol{\alpha}^{(2)} e^{\lambda t}. \tag{7.168}$$

Then a third solution corresponding to λ is of the form

$$(\boldsymbol{\alpha} t + \boldsymbol{\beta}) e^{\lambda t}, \tag{7.169}$$

where $\boldsymbol{\alpha}$ satisfies

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{\alpha} = \mathbf{0}, \tag{7.170}$$

and $\boldsymbol{\beta}$ satisfies

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{\beta} = \boldsymbol{\alpha}. \tag{7.171}$$

Now we must be careful here. Let us explain: Since $\boldsymbol{\alpha}^{(1)}$ and $\boldsymbol{\alpha}^{(2)}$ are both characteristic vectors corresponding to λ, both $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(2)}$ satisfy (7.170). However, in general, neither of these values of $\boldsymbol{\alpha}$ will be such that the resulting equation (7.171) in $\boldsymbol{\beta}$ will have a nontrivial solution for $\boldsymbol{\beta}$. Thus, instead of using the simple solutions $\boldsymbol{\alpha}^{(1)}$ or $\boldsymbol{\alpha}^{(2)}$ of (7.170), a more general solution of that equation is needed. Such a solution is provided by

$$\boldsymbol{\alpha} = k_1 \boldsymbol{\alpha}^{(1)} + k_2 \boldsymbol{\alpha}^{(2)}, \tag{7.172}$$

where k_1 and k_2 are suitable constants. We now substitute (7.172) for $\boldsymbol{\alpha}$ in (7.171) and determine k_1 and k_2 so that the resulting equation in $\boldsymbol{\beta}$ *will have* a nontrivial solution for

$\boldsymbol{\beta}$. With these values chosen for k_1 and k_2, we thus have the required $\boldsymbol{\alpha}$ and now find the desired nontrivial $\boldsymbol{\beta}$. The three resulting solutions (7.168) and (7.169) thus determined are linearly independent. We illustrate this situation in the following example.

EXAMPLE 3:

Consider the homogeneous linear system

$$\frac{dx_1}{dt} = 4x_1 + 3x_2 + x_3,$$

$$\frac{dx_2}{dt} = -4x_1 - 4x_2 - 2x_3, \tag{7.173}$$

$$\frac{dx_3}{dt} = 8x_1 + 12x_2 + 6x_3,$$

or in matrix form,

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{pmatrix}\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{7.174}$$

Assuming a solution of the form

$$\mathbf{x} = \alpha e^{\lambda t},$$

that is,

$$x_1 = \alpha_1 e^{\lambda t}, \quad x_2 = \alpha_2 e^{\lambda t}, \quad x_3 = \alpha_3 e^{\lambda t},$$

we know that λ must be a solution of the characteristic equation of the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{pmatrix}.$$

This characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & 3 & 1 \\ -4 & -4-\lambda & -2 \\ 8 & 12 & 6-\lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, we see that the characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0,$$

the factored form of which is $(\lambda - 2)^3 = 0$. We thus see that the characteristic values of $\mathbf{A}$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = 2.$$

That is, the number 2 is a *triple* characteristic value of $\mathbf{A}$. We again use

$$\mathbf{A}\boldsymbol{\alpha} = \lambda\boldsymbol{\alpha} \tag{7.145}$$

to find the corresponding characteristic vector(s) $\boldsymbol{\alpha}$.

With $\lambda = 2$ and

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \tag{7.175}$$

(7.145) becomes

$$\begin{pmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 2\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Performing the indicated multiplications and equating corresponding components of this and then simplifying, we find that $\alpha_1, \alpha_2, \alpha_3$ must be a nontrivial solution of the system

$$\begin{aligned} 2\alpha_1 + 3\alpha_2 + \alpha_3 &= 0, \\ -4\alpha_1 - 6\alpha_2 - 2\alpha_3 &= 0, \\ 8\alpha_1 + 12\alpha_2 + 4\alpha_3 &= 0. \end{aligned}$$

Each of these three relationships is equivalent to each of the other two, and so the only relationship among $\alpha_1, \alpha_2, \alpha_3$ is that given most simply by

$$2\alpha_1 + 3\alpha_2 + \alpha_3 = 0. \tag{7.176}$$

Observe that

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = -2$$

and

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = -3$$

are two distinct solutions of relation (7.176). The corresponding vectors of the form (7.175) are thus

$$\boldsymbol{\alpha}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\alpha}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix},$$

respectively. Since each satisfies (7.145) with $\lambda = 2$, each is a characteristic vector corresponding to the triple characteristic value 2. Furthermore, it is easy to see that the two characteristic vectors $\boldsymbol{\alpha}^{(1)}$ and $\boldsymbol{\alpha}^{(2)}$ are linearly independent, whereas every set of three characteristic vectors corresponding to characteristic value 2 are linearly dependent. Thus the characteristic value $\lambda = 2$ of multiplicity $m = 3$ has the $p = 2$ linearly independent characteristic vectors

$$\boldsymbol{\alpha}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\alpha}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \tag{7.177}$$

corresponding to it. Hence this is an illustration of the situation described in the paragraph immediately preceding this example. Thus corresponding to the triple characteristic value $\lambda = 2$ there are two linearly independent solutions of system (7.173) of the form $\boldsymbol{\alpha} e^{\lambda t}$. These are $\boldsymbol{\alpha}^{(1)} e^{2t}$ and $\boldsymbol{\alpha}^{(2)} e^{2t}$, that is,

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} e^{2t} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} e^{2t},$$

or

$$\begin{pmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{pmatrix}, \tag{7.178}$$

respectively.

A third solution corresponding to $\lambda = 2$ is of the form

$$(\boldsymbol{\alpha} t + \boldsymbol{\beta}) e^{2t}, \tag{7.179}$$

where $\boldsymbol{\alpha}$ satisfies

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\alpha} = \mathbf{0} \tag{7.180}$$

and $\boldsymbol{\beta}$ satisfies

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\beta} = \boldsymbol{\alpha}. \tag{7.181}$$

Since both $\boldsymbol{\alpha}^{(1)}$ and $\boldsymbol{\alpha}^{(2)}$ given by (7.177) are characteristic vectors of $\mathbf{A}$ corresponding to $\lambda = 2$, they both satisfy (7.180). But, as noted in the paragraph immediately preceding this example, we need to use the more general solution

$$\boldsymbol{\alpha} = k_1 \boldsymbol{\alpha}^{(1)} + k_2 \boldsymbol{\alpha}^{(2)}$$

of (7.180) in order to obtain a nontrivial solution for $\boldsymbol{\beta}$ in (7.181).

Thus we let

$$\boldsymbol{\alpha} = k_1 \boldsymbol{\alpha}^{(1)} + k_2 \boldsymbol{\alpha}^{(2)} = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

and then (7.181) becomes

$$\begin{pmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{pmatrix}.$$

Performing the indicated multiplications and equating corresponding components of this, we obtain

$$\begin{aligned} 2\beta_1 + 3\beta_2 + \beta_3 &= k_1, \\ -4\beta_1 - 6\beta_2 - 2\beta_3 &= k_2, \\ 8\beta_1 + 12\beta_2 + 4\beta_3 &= -2k_1 - 3k_2. \end{aligned} \tag{7.182}$$

Observe that the left members of these three relations are all proportional to one another. Using any two of the relations, we find that $k_2 = -2k_1$. A simple nontrivial solution of this last relation is $k_1 = 1$, $k_2 = -2$. With this choice of k_1 and k_2, we find

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}; \tag{7.183}$$

and the relations (7.182) become

$$\begin{aligned} 2\beta_1 + 3\beta_2 + \beta_3 &= 1, \\ -4\beta_1 - 6\beta_2 - 2\beta_3 &= -2, \\ 8\beta_1 + 12\beta_2 + 4\beta_3 &= 4. \end{aligned}$$

Each of these is equivalent to

$$2\beta_1 + 3\beta_2 + \beta_3 = 1.$$

A nontrivial solution of this is

$$\beta_1 = \beta_2 = 0, \quad \beta_3 = 1;$$

and thus we obtain

$$\boldsymbol{\beta} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.184}$$

Therefore, with $\boldsymbol{\alpha}$ given by (7.183) and $\boldsymbol{\beta}$ given by (7.184), the third solution (7.179) is

$$\left[\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] e^{2t},$$

that is,

that is,

$$\begin{pmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{pmatrix} \tag{7.185}$$

The three solutions defined by (7.178) and (7.185) are linearly independent, and a general solution is the linear combination

$$c_1\begin{pmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{pmatrix} + c_2\begin{pmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{pmatrix} + c_3\begin{pmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{pmatrix}$$

of these three, where c_1, c_2, c_3 are arbitrary constants. That is, in component form, a general solution of system (7.173) is

$$\begin{aligned} x_1 &= c_1e^{2t} + c_3te^{2t}, \\ x_2 &= c_2e^{2t} - 2c_3te^{2t}, \\ x_3 &= -2c_1e^{2t} - 3c_2e^{2t} + c_3(4t+1)e^{2t}, \end{aligned}$$

where c_1, c_2, c_3 are arbitrary constants.

Exercises

Find the general solution of each of the homogeneous linear systems in Exercises 1–24, where in each exercise

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

1. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix}\mathbf{x}.$

2. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{pmatrix}\mathbf{x}.$

3. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}\mathbf{x}.$

4. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{pmatrix}\mathbf{x}.$

5. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}\mathbf{x}.$

6. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\mathbf{x}.$

7. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{x}.$

8. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & 3 & -3 \\ -3 & 5 & -3 \\ 3 & 3 & -7 \end{pmatrix}\mathbf{x}.$

9. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 7 & 0 & 4 \\ 8 & 3 & 8 \\ -8 & 0 & -5 \end{pmatrix}\mathbf{x}.$

10. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{7}{5} & \frac{2}{5} & -\frac{1}{5} \\ \frac{4}{5} & \frac{9}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{4}{5} \end{pmatrix}\mathbf{x}.$

11. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -3 & 9 \\ 0 & -5 & 18 \\ 0 & -3 & 10 \end{pmatrix}\mathbf{x}.$

12. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & \frac{2}{7} & -\frac{4}{7} \\ 0 & \frac{19}{7} & \frac{4}{7} \\ 0 & \frac{6}{7} & \frac{9}{7} \end{pmatrix}\mathbf{x}.$

13. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 11 & 6 & 18 \\ 9 & 8 & 18 \\ -9 & -6 & -16 \end{pmatrix}\mathbf{x}.$

14. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 9 & 9 \\ 0 & 19 & 18 \\ 0 & 9 & 10 \end{pmatrix}\mathbf{x}.$

15. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{pmatrix}\mathbf{x}.$

16. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{pmatrix}\mathbf{x}.$

17. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -5 & -3 & -3 \\ 8 & 5 & 7 \\ -2 & -1 & -3 \end{pmatrix}\mathbf{x}.$

18. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 2 & 1 \\ -4 & -3 & -4 \\ 1 & 1 & 4 \end{pmatrix}\mathbf{x}.$

19. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 7 & 4 & 4 \\ -6 & -4 & -7 \\ -2 & -1 & 2 \end{pmatrix}\mathbf{x}.$

20. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 7 & 9 \\ -4 & -4 & -7 \end{pmatrix}\mathbf{x}.$

21. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{pmatrix}\mathbf{x}.$

22. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\mathbf{x}.$

23. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{pmatrix}\mathbf{x}.$

24. $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 6 & -1 \\ -1 & -2 & 1 \\ -2 & -8 & 4 \end{pmatrix}\mathbf{x}.$